"GEOMETRY OF WARPED PRODUCT SUBMANIFOLDS OF KAEHLERIAN MANIFOLDS"

DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

IN

MATHEMATICS

By

KAMRAN KHAN

Under The Supervision Of

DR. VIQAR AZAM KHAN

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
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Dedicated

to my

Beloved Parents

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<u>Certificate</u>

This is to certify that the contents of this dissertation entitled "Geometry of Warped Product Submanifolds of Kaehlerian Manifolds" has been written by Mr. Kamran Khan under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh as a partial fulfilment for the award of the degree of Master of Philosophy in Mathematics. To the best of my knowledge, the exposition has not been submitted to any other university/institution.

I further certify that Mr. Kamran Khan has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University, Aligarh.

CHAIRMAN
DEPARTMENT OF MATHEMATICS
A.M.U., ALIGARH

(Dr. V.A. Khan)

Supervisor

CONTENTS

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Kamran Khan

KAMRAN KHAN

PREFACE

The theory of submanifolds of an almost Hermitian manifold has been one of the most interesting topics in differential geometry. According to the behavior of the tangent bundle of a submanifold with respect to the almost complex structure J of the ambient manifold, there are two well known classes of submanifolds, the complex submanifolds and the totally real submanifolds. The study of complex submanifolds of a Kaehlerian manifold with a differential geometric point of view (that is, with emphasis on the Riemannian metric) was initiated by E. Calabi and others in the early 1950's (cf. [10]). On the other hand, the study of totally real submanifolds from the differentiable geometric point of view was initiated in the early 1970's (cf. [12],[13],[41],[58] etc.). Since then many differential geometers have contributed many interesting results in this subject. A. Bejancu [2] provided a single setting to study these submanifolds by introducing CR-submanifolds of Kaehler manifolds. CR-product submanifolds are defined as the Riemannian products of a holomorphic and totally real submanifolds of an almost Hermitian manifold and therefore can be assumed to be a special case of CR-submanifolds. B. Y. Chen [15] obtained various conditions under which a CR-submanifold of a Kaehler manifold reduces to a CR-product submanifold.

A more general class of product manifolds, known as warped product manifold was earlier defined by Bishop and O'Neill [6] while constructing example of manifolds of negative curvatures. These manifolds appear in differential geometric studies in a natural way and form the main theme of the dissertation. A warped product of the manifolds M_1 and M_2 is denoted by $M_1 \times_f M_2$, where f is a differentiable function on M_1. Ever since S. Nölker [47] gave an explicit description of the warped product representation of the Euclidean space, there have been studies of warped product spaces with extrinsic geometric point of views. However more recently the studies of the warped product manifolds with extrinsic geometric point of view are intensified after the impulse given by B.Y. Chen when he initiated the study of CR-submanifolds as warped products in a Kaehler manifold (cf. [18], [19]). In view of the various applications of warped product manifolds, some other warped product manifolds in different settings were later investigated by V.Bonanzinga and K. Matsumoto [9], V. Khan et. al [38] and , N. Jamal et. al [32]. The present dissertation gives an account of these works and the related problems in this area.

The dissertation comprises of five chapters. Each chapter is divided into various sections. The Mathematical relations obtained in the text have been labeled with double decimal numberings. The first figure denotes the chapter number, second represents the sections and the third points out the number of Theorem, Lemma, Proposition, Definition or the equation as the case may be, for example Theorem 3.2.4 refers to the fourth theorem of section 2 in the third Chapter.

The first chapter is introductory and serves the purpose of fixing the notations and developing the basic concepts keeping in view of the pre-requisites of the subsequent chapters and also to make the dissertation self contained.

As the theme of the dissertation is warped product submanifolds in various ambient manifolds, we have collected some basic results on the geometry of CR and generic submanifolds of Kaehler manifolds in Chapter 2. Integrability conditions of the canonical distributions on a generic and CR-submanifold of Kaehler manifold are discussed and the geometry of the leaves of the distributions are studied. The relevant results from the paper of B.Y. Chen [15] are incorporated in the chapter.

Chapter 3 is devoted to the study of warped product manifolds with intrinsic and extrinsic point of view. In fact, Bishop and O'Neill [6] in their initial paper found various fundamentally important geometric properties of warped product manifolds with intrinsic geometric point of view. These properties play important role in exploring even extrinsic differential geometric properties of warped product manifolds in various ambient manifolds. B.Y. Chen [18] initiated the study of warped product submanifolds of a Kaehler manifold. That is, submanifolds of the type $N_\perp \times_f N_T$ and $N_T \times_f N_\perp$ in a Kaehler manifold $\bar{M}$ where N_T and $N_\perp$ are respectively holomorphic and totally real submanifolds of $\bar{M}$. He proved that the non-trivial warped product submanifolds of the type $N_\perp \times_f N_T$ are non-existent in Kaehler manifolds whereas he and others found many examples of warped product submanifolds of the type $N_T \times_f N_\perp$ in Kaehler manifolds. An estimate of the squared norm of the second fundamental form is obtained for the CR-warped product submanifolds in Kaehler manifolds. B. Sahin [49] extended the study to semi-slant warped product submanifolds of Kaehler manifolds, namely the submanifolds of the type $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$ where N_θ is a proper slant submanifold of the underlying Kaehler manifold. He showed that these submanifolds are also non-existent. An account of these studies is presented in Chapter 3.

Chapter 4 is devoted to the study of warped product submanifolds of nearly

Kaehler manifolds. V.A. Khan et. al [36] established the non-existence of warped product CR-submanifolds. With regard to the extrinsic geometric features of a CR-warped product submanifolds of a nearly Kaehler manifold, an estimate for the squared norm of the second fundamental form is obtained in Chapter 4 (cf. [39]).

The study of warped product submanifolds is extended to the setting of locally conformal Kaehler manifold by K. Matsumoto [42] . Chapter 5 is devoted to the study of warped product submanifolds of l.c.K. manifolds. As a step forward, generic warped product submanifolds of l.c.K. manifolds are considered by N. Jamal and V.A. Khan [32]. They devised the method to construct generic warped product submanifolds of l.c.K. manifolds. The results are strengthened by an example. All these investigations are presented in Chapter 5.

In the end of the dissertation, references have been given which by no means are comprehensive but mention only the papers and books referred to in the main body of the dissertation.

CHAPTER 1

INTRODUCTION

The aim of this chapter is to introduce basic concepts, preliminary notions and some fundamental results that are required for the development of the subject in the present dissertation. In this chapter we have given a brief resume of some of the results in the geometry of almost Hermitian manifolds, some allied structures and the geometry of submanifolds of these manifolds. Although most of these results are readily available in review articles and some in standard books, e.g., Nomizu and Kobayashi [40], B.Y.Chen [11], Yano and Kon [60] etc., nevertheless we have collected them here to fix up our terminology and for ready references.

1.1. Smooth Manifolds

A differentiable manifold is, roughly speaking, a space which locally looks like $\mathbb{R}^n$ and on which we can talk of C^k-functions for $1 \leq k \leq \infty$. Formally, we can define manifold as follows

Definition 1.1.1. Let M be a set. Assume that there exists a collection of subsets $\{U_\alpha : \alpha \in A\}$ on M with the following properties

(i) $\bigcup\limits_{\alpha \in A} U_\alpha = M$

(ii) There exists a bijective map ϕ_α from U_α onto an open subset $\phi_\alpha(U_\alpha)$ of $\mathbb{R}^m$ for each $\alpha \in A$ with m a non-negative integer.

(iii) The following compatibility condition holds:

There exists a non-negative integer k such that the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is C^k, for all $\alpha, \beta \in A$, whenever $U_\alpha \cap U_\beta$ is non-empty.

The object $(M, \{(U_\alpha, \phi_\alpha) : \alpha \in A\})$ is called a C^k-*manifold.* The collection $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ is called a C^k-*atlas* on M. The members of the atlas are called *charts* or *co-ordinate charts.* They are called co-ordinate charts since they give a system of local co-ordinates. An atlas is said to be *smooth* if it is a C^k-atlas for all $k \in N$. A manifold equipped with a smooth atlas is called a *differentiable* or *smooth manifold.* The integer m is called dimension of M.

Following are some examples of differentiable manifolds.

Example 1.1.1. The most common example of a differentiable manifold is $\mathbb{R}^n$. If $M = \mathbb{R}^n, U = M$ and ϕ is the identity map, then (U, ϕ) is a co-ordinate system covering $\mathbb{R}^n$.

Example 1.1.2. The circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a differentiable manifold of dimension 1.

Example 1.1.3. The two dimensional torus $T^2 = S^1 \times S^1$ is a 2-dimensional differentiable manifold.

Example 1.1.4. The sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a 2-dimensional differentiable manifold.

Definition 1.1.2. Let M be a differentiable manifold with dimension m and $p \in M$. We denote by $\mathcal{F}(p)$ the algebra of differentiable functions defined in a neighbourhood of p. Let $\gamma(t), (a \leq t \leq b)$ be a curve passing through p with $\gamma(t_0) = p$. The vector tangent to the curve $\gamma(t)$ at p is a mapping $X_p : \mathcal{F}(p) \longrightarrow \mathbb{R}$ defined by

$$X_p(f) = \frac{d}{dt}(f \circ \gamma) \mid_{t=t_0} \tag{1.1.1}$$

The mapping X_p satisfies the following conditions

(*i*) $X_p(af + bg) = aX_pf + bX_pg$

(*ii*) $X_p(f.g) = f(p).X_p(g) + g(p).X_p(f)$

for all $f, g \in \mathcal{F}(p)$. In general, a tangent vector on a manifold at a point p is characterized by the properties (i) and (ii). The set of all tangent vectors at a point $p \in M$, is a real vector space of dimension m, called *tangent space* and is denoted by $T_p(M)$.

Let (U, ϕ) be a co-ordinate chart with a co-ordinate system (x^i) and $p \in U$

such that $\phi(p) = 0$ (such co-ordinate system exists). For any $f \in \mathcal{F}(p)$, let

$$\frac{\partial}{\partial(x^i)} \mid_p (f) = \frac{\partial}{\partial(u^i)}(f \circ \phi^{-1}) \mid_0 \tag{1.1.2}$$

where (u^i) are the natural co-ordinates on $\mathbb{R}^m$. It is easy to check that $\frac{\partial}{\partial(x^i)} \mid_p$ is a tangent vector at p.

A vector field X on a manifold M, is a correspondence that associates to each point $p \in M$ a tangent vector X_p at p. Locally, we can express a vector field as

$$X = \xi^i(x)\frac{\partial}{\partial(x^i)}$$

on a neighbourhood with a local co-ordinate system $(x^i), 1 \leq i \leq m$ where $\xi^i(x)$ are differentiable functions on M. Hence, a vector field can be regarded as a derivation on the algebra of smooth functions on M. The space of all smooth vector fields on a smooth manifold M forms a Lie-algebra with Lie-product of vector fields X and Y, defined as

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf) \tag{1.1.3}$$

Definition 1.1.3. Let α be a co-variant tensor of order p (set of all covariant tensors of order p is denoted by $((\otimes)^p T^*_M)$ and S_p, the permutation group of p integers $(1, 2, ..., p)$. Then $\pi \in S_p$ is a linear mapping of $((\otimes)^p T^*_M)$ into itself defined by the relation

$$(\pi\alpha)(X_1, ..., X_p) = \alpha(X_{\pi(1)}, ..., X_{\pi(p)})$$

α is symmetric iff $\pi\alpha = \alpha$, and α is completely skew-symmetric or *alternating* if $\pi\alpha =$ (sign π) α where sign $\pi = +1$ or -1 according as π is even or odd permutation.

A *p-form* (exterior differentiable form of degree p) is a field of alternating p-cotensors.

Definition 1.1.4. A *distribution* D of dimension r on a manifold M is an assignment to each point p of M, an r-dimensional subspace D_p of $T_p(M)$. It is called differentible if every point p has a neighbourhood U and r-differentible vector fields on U, say $X_1, X_2, ..., X_r$ which form a basis of D_q at every $q \in U$. The set $X_1, X_2, ..., X_r$ is called *a local basis* of D in U. A vector field X is said to belong to D if $X_p \in D_p$ for all $p \in M$. D is called *involutive* if $[X, Y] \in D$ whenever two vector fields $X, Y \in D$.

A connected submanifold N of M is called *integral manifold* of the distribution D if $f_*(T_pN) = D_p$ for all $p \in N$, where f is the embedding of N into M. If there is no other integral manifold of D which contains N, then N is called a *maximal integral manifold* of D. The classical theorem of Frobenius can be formaluted as follows

Theorem 1.1.1. [40] *Let D be an involutive distribution on a manifold M. Through every point p $\in$ M, there passes a unique maximal integral manifold N(p) of D. Any integral manifold through p is an open submanifold of N(p).*

1.2. Riemannian Manifolds and Submanifolds of a Riemannian Manifold

To study the geometry of a smooth manifold, we initially need to have a Riemannian metric on it, i.e., a positive definite inner product on the tangent bundle TM of the Riemannian manifold M. To be more precise, by a Riemannian metric g on a manifold M, we mean a map: $p \longrightarrow g_p$, where g_p is a positive definite inner product on $T_p(M)$. We require this map to be smooth in the sense that the function

$$p \longrightarrow g_{ij}(p) = g_p(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p)$$

is smooth for all i, j on any chart (U, p). This smoothness condition is same as requiring that for all vector fields X, Y on M, the map

$$p \longrightarrow g_p(X_p, Y_p)$$

is smooth. On a paracompact manifold, there exists a Riemannian metric. A smooth manifold with a Riemannian metric is said to be a *Riemannian manifold.*

Fundamental Theorem of Riemannian Geometry[60]. On every Riemannian manifold (M, g) there exist a unique connection ∇ satisfying the following conditions:

(i) The torsion T vanishes, i.e.

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

(ii) g is parallel, i.e., $(\nabla_X g) = 0$.

The connection ∇ is called *Riemannian connection* (or, *Levi-Civita connection*). The Levi-Civita connection of a manifold M is characterized by Koszul formula

$$
\begin{aligned}
2g(\nabla_X Y, Z) = & \, Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\
& - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])
\end{aligned}
$$

for any vector fields X, Y and Z on M.

The curvature tensor R on a manifold M with connection ∇ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

We now define

$$K(X,Y;Z,W) = g(R(X,Y)Z,W)$$

which is a (0,4) tensor and is called the *Riemannian curvature tensor.*

Let X and Y be two linearly independent vectors at a point p and $\gamma(X,Y)$ be the plane section spanned by X and Y. The *sectional curvature* $k(\gamma)$ for γ is defined by

$$k(\gamma) = \frac{K(X,Y;X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{1.2.1}$$

It is easy to see that the $k(\gamma)$ is uniquely determined by the plane section γ and is independent of the choice of X and Y on it.

If $k(\gamma)$ is constant for all plane section γ in the tangent space $T_p(M)$ at p and for all points $p \in M$, then M is called a *space of constant curvature.*

Theorem 1.2.1. *Let M be a Riemannian manifold of dimension $n > 2$. If the sectional curvature $k(\gamma)$ depends only on the point p, then M is a space of constant curvature.*

A Riemannian manifold of constant sectional curvature is known as a *real space form.* The curvature tensor R of a real space form of sectional curvature c is given by

$$R(X,Y)Z = c\{g(Z,Y)X - g(Z,X)Y\}$$

for any vector fields X, Y and Z tangent to the real space form.

A Riemannian manifold of constant curvature is said to be *elliptic, hyperbolic* or *flat* (or *locally Euclidean*) according as the sectional curvature is positive, negative or zero. These spaces are called *space forms.*

If an n-dimensional differentiable manifold M admits an immersion

$$f : M \longrightarrow \bar{M}$$

into an m-dimensional manifold $\bar{M}$, then M is said to be a *submanifold* of $\bar{M}$. Naturally $n \leq m$. If $\bar{M}$ is a Riemannian manifold with a Riemannian metric g, then M admits a Riemannian metric induced from $\bar{M}$ which we will denote by the same symbol g. The immersion f is said to be an *isometric immersion* if the differentiable map

$$f_* : TM \longrightarrow T\bar{M}$$

preserves the Riemannian metric, that is for any $X, Y \in TM$,

$$g(f_*X, f_*Y) = g(X, Y) \tag{1.2.2}$$

When only local questions are involved, we shall identify TM with $f_*(TM)$ through the isomorphism f_*. Hence, a tangent vector in $T\bar{M}$ tangent to M, shall mean a tangent vector which is the image of an element in TM under f_*. More generally, a C^∞-cross section of the restriction of $T\bar{M}$ on M, shall be called a vector field of $\bar{M}$ on M. Those tangent vectors of $T\bar{M}$ which are normal to TM form the normal bundle $T^\perp M$ of M. Hence, for every point $p \in M$, the tangent space $T_{f(p)}(\bar{M})$ admits the following decomposition

$$T_{f(p)}(\bar{M}) = T_p(M) \oplus T_p^\perp(M)$$

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^\perp$ on TM and $T^\perp M$ respectively governed by Gauss and Weingarten formulae viz.

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.2.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{1.2.4}$$

for $X, Y \in TM$ and $N \in T^\perp M$. h and A are the second fundamental form and the shape operator respectively. They are related as

$$g(h(X, Y), N) = g(A_N X, Y) \tag{1.2.5}$$

Looking into Gauss and Weingarten formulae, one can classify the submanifolds, putting conditions on h as follows:

Definition 1.2.1. A submanifold for which the second fundamental form h is identically zero is called a *totally geodesic submanifold.*

Definition 1.2.2. A submanifold is called a *totally umbilical submanifold* if its second fundamental form h satisfies

$$h(X,Y) = g(X,Y)H \tag{1.2.6}$$

for any $X, Y \in TM$ where

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

is the *mean curvature vector*, where $(e_i)_{1\leq i\leq n}$ is a local orthonormal basis in $T\bar{M}$. The squared norm of the second fundamental form h is defined by

$$\| h \|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \tag{1.2.7}$$

Definition 1.2.3. A submanifold is called a *minimal submanifold* if the mean curvature vector vanishes identically i.e., $H = 0$.

For the second fundamental form h, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection $\bar{\nabla}$ in $T\bar{M}$ by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1.2.8}$$

for $X, Y, Z \in TM$.

Let $\bar{R}$ and R denote the curvature tensors of the connections on $\bar{M}$ and M respectively, then the equation of Gauss, Codazzi and Ricci are given by

$$R(X,Y;Z,W) = \bar{R}(X,Y;Z,W) + g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)) \tag{1.2.9}$$

$$[\bar{R}(X,Y), Z]^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{1.2.10}$$

$$(1.2.11) \qquad \bar{R}(X,Y;N_1N_2) = R^{\perp}(X,Y;N_1,N_2) - g([A_{N_1},A_{N_2}]X,Y)$$

for $X, Y, Z, W \in TM$ and $N_1, N_2 \in T^{\perp}M$, where $\perp$ denotes the normal component.

A submanifold M of $\bar{M}$ is called *auto-parallel* if for each $X \in T_p(M)$ and for each curve γ in M starting from p, the parallel displacement of X along γ (with respect to the affine connection $\bar{\nabla}$of $\bar{M}$) yields a vector field tangent to M. Thus, a distribution D on a manifold M is *parallel* if $\nabla_X Y \in D$ for each $X, Y \in D$. It is straightforward to observe that a parallel distribution is integrable and its leaves are totally geodesic in M.

Note 1.2.1. Let (M, g) be a Riemannian manifold and D^0 denote the orthogonal complement of a distribution D on M. Then it is easy to observe that if D is M-parallel then both D and D^0 are parallel. In other words the leaves of D and D^0 are totally geodesic in M. That is, M is the Riemannian product of the leaves of D and D^0.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with Riemannian metric g_1 and g_2 respectively. Then the product manifold $M = M_1 \times M_2$ is a Riemannian manifold endowed with the Riemannian metric g defined as

$$(1.2.12) \qquad g(U,V) = g_1(d\pi_1 U, d\pi_1 V) + g_2(d\pi_2 U, d\pi_2 V)$$

where $d\pi_i (i = 1, 2)$ are the differentials of the projections $\pi_i : M_1 \times M_2 \to M_i$.

R.L. Bishop and B. O'Neill [6] introduced the notion of warped product manifolds while investigating manifolds of negative curvatures. These manifolds appear as a generalized version of Riemannian product of manifolds. In fact, the warped product of two manifolds M_1 and M_2 is obtained by homothetically warping the product metric on to the fibers $x \times M_2$ for each $x \in M_1$. Formally speaking the warped product manifolds are defined as

Definition 1.2.4. [6] Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifold with Riemannian metric g_1 and g_2 respectively and f a positive differentiable function on M_1. The warped product $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with the Riemannian metric g defined as

$$(1.2.13) \qquad g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2).$$

More explicitly, if U is tangent to $M = M_1 \times_f M_2$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2,$$

the function f is known as the *warping function.*

A warped product manifold is called a *trivial warped product* if the warping function f is constant. In other words, a trivial warped product $M_1 \times_f M_2$ is nothing but a Riemannian product $M_1 \times M_2^f$, where M_2^f is the Riemannian manifold with Riemannian metric $f^2 g_2$ which is homothetic to original metric g_2 of M_2.

Example 1.2.1. A surface of revolution is a warped product with leaves the different positions of the rotated curve and fibres the circles of revolution. More explicitly, if M is obtained by revolving a plane curve C about an axis in $\mathbb{R}^3$ and $f : C \longrightarrow \mathbb{R}^+$ gives distance to the axis, $M = C \times_f S^1(1)$ is a warped product manifold. Here $S^1(1)$ denotes the unit circle.

Example 1.2.2. In spherical co-ordinates, the line element of $\mathbb{R}^3 - \{0\}$ is

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Setting $r = 1$ gives the line element of unit sphere S^2. Evidently $\mathbb{R}^3 - \{0\}$ is diffeomorphic to $\mathbb{R}^+ \times S^2$ under the natural map $(t,p) \longleftrightarrow tp$. Thus the formula for ds^2 shows that $\mathbb{R}^3 - \{0\}$ can be identified with the warped product $\mathbb{R}^+ \times_r S^2$. In $\mathbb{R}^3 - \{0\}$ the leaves are the rays from the origin and the fibres are the spheres $S^2(r), r > 0$. In general, $\mathbb{R}^n - \{0\}$ is naturally isomorphic to $\mathbb{R}^+ \times_r S^{n-1}$.

Example 1.2.3. The standard space-time models of the universe are warped products, as are the simplest models of neighbourhoods of stars and black holes.

Every Riemannian manifold of constant scalar curvature c can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For example $S^n(1)$ is locally isometric to $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{cost} S^{n-1}$, $\mathbb{R}^n$ is locally isometric to $(0, \infty) \times_x S^{n-1}$, $H^{n-1}(-1)$ is locally isometric to $\mathbb{R} \times_{e^x} \mathbb{R}^{n-1}$.

R. L. Bishop and B. O'Neill [6] obtained the following results for a warped product manifold which we have used extensively in our subsequent analysis of the subject.

Proposition 1.2.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in TM_1$ and $Z, W \in TM_2$, then* *(i)* $\nabla_X Y \in TM_1$,

(ii) $\nabla_X Z = \nabla_Z X = (Xlnf)Z$,

(iii) $nor(\nabla_Z W) = -g(Z,W)\nabla lnf$

where ∇ is the Riemannian connection on M and $nor(\nabla_Z W)$ denotes the component of $\nabla_Z W$ in TM_1 and ∇f (or grad f) is defined as $g(\nabla f, X) = Xf$.

Corollary 1.2.1. *In a warped product manifold $M = M_1 \times_f M_2$,*

(i) M_1 *is totally geodesic in* M

(ii) M_2 *is totally umbilical in* M.

We have the following result of S.Hiepko [31] which is applied by many authors to ensure the existence of a warped product submanifold in a given ambient manifold.

Theorem 1.2.2. *Let F be a vector subbundle in the tangent bundle of a Riemannian manifold M and let $F^\perp$ be its normal bundle. Assume that the two distributions are both involutive and the integral manifolds of F (resp. $F^\perp$) are totally geodesic (resp. extrinsic spheres). Then M is locally isometric to a warped product $M_1 \times_f M_2$, where M_1 and M_2 are the leaves of F and $F^\perp$ respectively. Moreover, if M is simply connected and complete, there exists a global isometry of M with a warped product.*

1.3. Structures on Manifolds

Further refined information can be had by knowing additional structures on the manifold, for example almost complex, almost Hermitian, Kaehler, nearly Kaehler, locally conformal Kaehler and almost contact structures [40],[60]. In this section, we briefly discuss some of these structures.

In what follows, we shall take a differentiable manifold M that is connected and paracompact, so that it can always be endowed with a Riemannian metric g and a Riemannian connection ∇.

Given a smooth real valued function f on a Riemannian manifold M endowed with a Riemannian metric g and the Levi-Civita connection ∇, the gradient and Laplacian of f, denoted respectively as ∇f and $\triangle f$, are defined by

(1.3.1) $$g(\nabla f, U) = Uf,$$

and

(1.3.2) $$\triangle f = \sum_{j=1}^{n}\{e_j(e_j f) - (\nabla_{e_j} e_j)f\}$$

for each $U \in TM$, where TM is the tangent bundle of M and $\{e_1, e_2, ..., e_n\}$ is a local frame of orthonormal vector fields on M.

An *almost complex structure* on a real differentiable manifold M is a tensor field J which is at every point $p \in M$, an endomorphism of the tangent space $T_p(M)$ i.e., $J_p : T_pM \longrightarrow T_pM$ such that $J_p^2 = -I$ where I denotes the identity transformation on $T_p(M)$. A manifold with a fixed almost complex structure is called *an almost complex manifold.* On a paracompact almost complex manifold, there is always a Riemannian metric g consistent with the almost complex structure J satisfying

(1.3.3) $$g(JX, JY) = g(X, Y)$$

for all $X, Y \in TM$, by virtue of which g is called a *Hermitian metric.* An almost complex (resp. complex) manifold with a Hermitian metric is called an *almost Hermitian* (resp. *Hermitian*) manifold.

The fundamental 2-form Ω of an almost Hermitian manifold M with an almost complex structure J and metric g is defined by

(1.3.4) $$\Omega(X, Y) = g(JX, Y)$$

since g is invariant by J, so is Ω, i.e.,

(1.3.5) $$\Omega(JX, JY) = \Omega(X, Y)$$

for all vector fields X, Y in TM.

The almost complex structure J is not, in general, parallel with respect to the Riemannian connection defined by the Hermitian metric g. In fact, we have the following formula

(1.3.6) $$4g((\nabla_X J)Y, Z) = 6d\Omega(X, JY, JZ) - 6d\Omega(X, Y, Z) + g(N(Y, Z), JX)$$

for any vector fields X, Y, Z on M, where N denotes the Nijenhuis tensor of J. In general, the Nijenhuis tensor field of a (1,1)-tensor A on a manifold M is a (1,2)-tensor field defined by

$$N(X,Y) = [AX, AY] + A^2[X,Y] - A[X, AY] - A[AX, Y] \tag{1.3.7}$$

In particular, the Nijenhuis tensor of J is given by

$$N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y] \tag{1.3.8}$$

It is easy to verify that the Nijenhuis tensor of J satisfies

$$N(JX,Y) = N(X,JY) = -JN(X,Y) \tag{1.3.9}$$

for all vector fields X, Y on M. The vanishing of the Nijenhuis tensor N of J is the necessary and sufficient condition for an almost complex manifold to be a complex manifold.

If we extend the Riemannian connection ∇ to be a derivative on the tensor algebra of M, then we have the following formula

$$(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y \tag{1.3.10}$$

Definition 1.3.1. A Hermitian metric on an almost complex manifold is called a *Kaehler metric* if the fundamental 2-form Ω is closed. A complex manifold equipped with a Kaehler metric is said to be a *Kaehler manifold.* Thus by formula (1.3.6), an almost complex manifold $\bar{M}$ is Kaehler if and only if

$$(\bar{\nabla}_X J)Y = 0 \tag{1.3.11}$$

for all $X, Y \in T\bar{M}$. In this case, the connection $\bar{\nabla}$ on $\bar{M}$ is said to be the *Kaehlerian connection.*

There is a more general condition than the condition (1.3.11) namely

$$(\bar{\nabla}_X J)X = 0 \tag{1.3.12}$$

for all vector fields X. An almost Hermitian manifold characterized by the above condition is said to be a *nearly Kaehler* manifold. The unit 6-sphere admits a nearly Kaehler structure described in the following example.

Example 1.3.1. [24]. Let S^6 be the 6-dimensional unit sphere defined as follows. Let E^7 be the set of all purely imaginary Cayley numbers. Then E^7 is the 7-dimensional subspace of Cayley algebra C. Let $\{1 = e_0, e_1, ..., e_7\}$ be a basis of the Cayley algebra, 1 being the unit element of C. If $X = \sum_{i=1}^{7} x^i e_i$ and $Y = \sum_{i=1}^{7} y^i e_i$ are two elements of E^7, one defines the *scaler product* in E^7 by

$$g(X,Y) = \sum_{i=1}^{7} x^i y^i$$

and the *vector product* by

$$X \times Y = \sum_{i \neq j} x^i y^i e_i \star e_j,$$

$\star$ being the multiplication operation of C given in the following table:

$e_i \times e_j =$

i/j	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

Consider the 6-dimensional unit sphere S^6 in E^7:

$$S^6 = \{X \in E^7 : g(X,X) = 1\}$$

The scaler product in E^7 induces the natural metric tensor field g on S^6. The tangent space $T_X S^6$ at $X \in S^6$ can naturally be identified with the subspace of E^7 orthogonal to X. Define the endomorphism J_X on $T_X S^6$ by

$$J_X Y = X \times Y, \qquad for \qquad Y \in T_X S^6$$

It is easy to see that

$$g(J_X Y, J_X Z) = g(Y, Z), \qquad Y, Z \in T_X S^6$$

The correspondence $X \longrightarrow J_X$ defines a tensor field J such that $J^2 = -I$. Consequently, S^6 admits an almost Hermitian stucture (J, g). This structure is a non-Kaehler nearly-Kaehlerian structure.

Lemma 1.3.1. [60] *On a nearly Kaehler manifold $\bar{M}$, we have*

(i) $(\bar{\nabla}_{JV} J)U = J(\bar{\nabla}_U J)V$;

(ii) $N(U, V) = -8J(\bar{\nabla}_U J)V$,

for all vector fields U, V on $\bar{M}$.

Definition 1.3.2. An almost Hermitian manifold $(\bar{M}, J, g)$ is called a *locally conformal Kaehler* (briefly l.c.K.) manifold if for any $x \in \bar{M}$, there is an open neighbourhood U such that for some differentiable function $\lambda : U \longrightarrow \mathbb{R}$, the *conformal metric* $g' = e^{-\lambda} g \mid_U$ is a Kaehler metric on U i.e., $\bar{\nabla}(e^{-\lambda} J) = 0$ where $\bar{\nabla}$ denotes the covariant differentiation with respect to g'. If $U = \bar{M}$, then the manifold $\bar{M}$ is called a *globally conformal Kaehler* (briefly g.c.K.) manifold.

It is known that $\bar{M}$ is l.c.K. if there is a closed 1-form α globally defined on $\bar{M}$, such that $d\Omega = \alpha \wedge \Omega$ (cf. [55]) where Ω is fundamental two-form on $\bar{M}$ associated with J and g. The closed 1-form α is called the Lee-form of the l.c.K. manifold. It is known that $(\bar{M}, J, g)$ is globally conformal Kaehler (respectively Kaehler) if the Lee-form α is exact (respectively $\alpha = 0$). Any simply connected l.c.K. manifold is g.c.K.

On an l.c.K. manifold, the Lee-vector field $\lambda = \alpha^\sharp$, where $\sharp$ means the raising of the indices with respect to g i.e., $g(U, \lambda) = \alpha(U)$ for all $U \in T(\bar{M})$. If $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$, then we have

$$(\bar{\nabla}_U J)V = \theta(V)U - \alpha(V)JU - g(U, V)\mu - \Omega(U, V)\lambda. \qquad (1.3.13)$$

where $\theta = \alpha \circ J$ and $\mu = -J\lambda$ are the anti-Lee form and the anti-Lee vector field, respectively [55]. In terms of the Lee-vector field, (1.3.13) can be written as

$$(1.3.14) \qquad (\bar{\nabla}_U J)V = g(\lambda, JV)U - g(\lambda, V)JU + g(JU, V)\lambda + g(U, V)J\lambda.$$

$S^{2n+1} \times S^1$ is a typical example of a compact locally conformal Kaehler manifold with parallel Lee-form [55]. We give a brief exposition of how the product manifold $S^{2n+1} \times S^1$ admits the structure of an l.c.K. manifold.

Example 1.3.2. Let $\mathbb{R}^{2n+2}$ be a $(2n+2)$-dimensional Euclidean space equipped with the canonical inner product g and $\{e_1, e_2, ..., e_{2n+1}, e_{2n+2}\}$ the canonical orthonormal basis of $\mathbb{R}^{2n+2}$. We denote by J_0 the complex structure on $\mathbb{R}^{2n+2}$ defined by

$$J_0 e_{2m-1} = e_{2m}, \quad J_0 e_{2m} = -e_{2m-1}, \quad 1 \leq m \leq n+1$$

Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere with the canonical Sasakian structure (ϕ, ξ, η, h) induced from the Kaehler structure (J_0, g) on $\mathbb{R}^{2n+2}$. It is well known that the structure vector field ξ defines the Hopf fibration $\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n$ where $\mathbb{C}P^n$ is an n-dimensional complex projective space equipped with the canonical Fubini study metric of constant holomorphic sectional curvature 4. Let $S^1 = \{e^{it}, t \in R \textit{ and } i = \sqrt{-1}\}$ be a unit circle. We define an almost complex structure J on $\bar{M} = S^{2n+1} \times S^1$ by

$$JT = \xi \qquad \textit{and} \qquad JU = \phi U,$$

for any vector fields U on $\bar{M}$ such that $\eta(U) = 0$, where $T = \frac{\partial}{\partial t}$ is the canonical unit vector field on S^1. Then $(S^{2n+1} \times S^1, J)$ is an l.c.K. manifold together with the product metric $g = h + 1$ on $\bar{M} = S^{2n+1} \times S^1$. The Lee form ω of $\bar{M}$ is given by $\omega = 2dt$.

An l.c.K. manifold $\bar{M}$ is said to be a *generalized Hopf manifold* if the Lee-form α is parallel, i.e., $\bar{\nabla}\alpha = 0$ on $\bar{M}$, because a Hopf manifold, which is diffeomorphic to $S^1 \times S^{2n-1}$.

An l.c.K. manifold $\bar{M}$ is called an *l.c.K. space form* if it has a constant holomorphic sectional curvature c. Then the Riemannian curvature tensor $\bar{R}$ of an l.c.K. space form with constant holomorphic sectional curvature c is given by

$$
\begin{aligned}
4\bar{R}(X,Y,Z,W) = {} & c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
& + g(JX,W)g(JY,Z) - g(JX,Z)g(JY,W) - 2g(JX,Y)g(JZ,W)\} \\
& + 3\{P(X,W)g(Y,Z) - P(X,Z)g(Y,W) + g(X,W)P(Y,Z) \\
& - g(X,Z)P(Y,W)\} - \bar{P}(X,W)g(JY,Z) + \bar{P}(X,Z)g(JY,W) \\
& - g(JX,W)\bar{P}(Y,Z) + g(JX,Z)\bar{P}(Y,W) \\
& + 2\{\bar{P}(X,Y)g(JZ,W) + g(JX,Y)\bar{P}(Z,W)\} \qquad (1.3.15)
\end{aligned}
$$

for any $X, Y, Z, W \in T\bar{M}$(cf. [45]).

If $k(\gamma)$ is a constant for all planes γ in $T_p(M)$ invariant by J and for all points $p \in M$, then M is called a *space of constant holomorphic sectional curvature*. The following is a Kaehlerian analogue of Schur's Theorem.

Theorem 1.3.1. *Let M be a connected Kaehler manifold of complex dimension $n \geq 2$. If the holomorphic sectional curvature $k(\gamma)$, where γ is a plane in $T_p(M)$ invariant by J, depends only on p, then M is a space of constant holomorphic sectional curvature.*

Definition 1.3.3. [40] A Kaehler manifold $\bar{M}$ is called a complex-space-form if it has constant holomorphic sectional curvature. We denote a complex-space-form of constant holomorphic sectional curvature c by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$
\begin{aligned}
\bar{R}(X,Y)Z = {} & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y \\
& + g(JY,Z)JX) - g(JX,Z)JY + 2g(X,JY)JZ\} \qquad (1.3.16)
\end{aligned}
$$

An *RK-manifold* $(\bar{M}, J, g, \bar{\nabla})$ is an almost Hermitian manifold for which the curvature tensor $\bar{R}$ is invariant under J, i.e.,

$$\bar{R}(JU, JV, JW, JZ) = \bar{R}(U, V, W, Z)$$

for any $U, V, Z, W \in T\bar{M}$

An almost Hermitian manifold $\bar{M}$ is of *pointwise constant type* if for any $p \in \bar{M}$ and $U \in T_p\bar{M}$

$$\lambda(U,V) = \lambda(U,W)$$

where $\lambda(U,V) = \bar{R}(U,V,JU,JV) - \bar{R}(U,V,U,V)$ with V and W being unit tangent vectors at p, orthogonal to U and JU. The manifold $\bar{M}$ is said to be of *constant type* if for any unit vectors $U, V \in T\bar{M}$ with $g(U,V) = g(JU,V) = 0, \lambda(U,V)$ is a constant function.

A *generalized complex space form* is an RK-manifold of constant holomorphic sectional curvature and of constant type. A generalized complex space form of constant holomorphic sectional curvature c and of constant type α is denoted by $\bar{M}(c,\alpha)$. Each complex space form is a generalized complex space form. The converse is not true. The sphere S^6 endowed with the standard nearly Kaehler structure is an example of a generalized complex space form which is not a complex space form.

The Riemannian curvature tensor $\bar{R}$ of $\bar{M}(c,\alpha)$ has the following expression

$$\begin{aligned}\bar{R}(U,V)W =& \frac{c+3\alpha}{4}[g(V,W)U - g(U,W)V] \\ &+ \frac{c-\alpha}{4}[g(U,JW)JV - g(V,JW)JU \\ &+ 2g(U,JV)JW]\end{aligned} \tag{1.3.17}$$

1.4. Submanifolds of an Almost Hermitian Manifold

The submanifolds of almost Hermitian manifolds have additional advantage because of the peculiar behaviour of the almost complex structure J which when acts on a vector transforms it into a vector perpendicular to the given vector. Let M be a submanifold of an almost Hermitian manifold $\bar{M}$. For any point $p \in M$, and $U \in T_pM$, decomposing JU into tangential and normal parts as

$$JU = PU + FU. \tag{1.4.1}$$

That is, $PU \in T_pM$ and $FU \in T_p^{\perp}M$. Then P is an endomorphism and F is a normal valued 1-form on T_pM. The 1-1 tensor field induced from the endomorphism P and the linear differential form induced from F are denoted by the same symbols P and F respectively.

Similarly, for any vector N normal to M, we put

$$JN = tN + fN, \tag{1.4.2}$$

with tN and fN as tangential and normal components of JN respectively then f is a 1-1 tensor field and t is a tangential valued 1-form on $T^{\perp}M$.

The covariant differentiation of the tensors P, F, t and f are defined respectively

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{1.4.3}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{1.4.4}$$

$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla^{\perp}_U N, \tag{1.4.5}$$

$$(\bar{\nabla}_U f)N = \nabla^{\perp}_U fN - f\nabla^{\perp}_U N, \tag{1.4.6}$$

Definition 1.4.1. [57] Let $(\bar{M}, J, g)$ be an almost Hermitian manifold. A submanifold M of $\bar{M}$ is called a *holomorphic* (or *invariant* or *almost complex*) submanifold of $\bar{M}$ if for any $p \in M$ the tangent space $T_p(M)$ is J-invariant, i.e. $J(T_p(M)) = T_p(M)$.

Definition 1.4.2. [58] Let $(\bar{M}, J, g)$ be an almost Hermitian manifold. A submanifold M isometrically immersed in $\bar{M}$ is called an *anti-invariant submanifold* of $\bar{M}$ (or *totally real submanifold* of $\bar{M}$) if $JT_p(M) \subset T_p^{\perp}(M)$ for each point $p \in M$.

Definition 1.4.3. [2] Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and Hermitian metric g, then a submanifold M of $\bar{M}$ is said to be a *Cauchy Riemannian submanifold* (or in short a *CR-submanifold*) if there exists a differentiable distribution $D : p \longrightarrow D_p \subset T_p(M), (p \in M)$ on M satisfying the following conditions:

(i) D is invariant i.e. $JD_p = D_p$ for all $p \in M$.

(ii) The complementary distribution $D^{\perp} : p \longrightarrow D_p^{\perp} \subset T_p(M), p \in M$ is anti-invariant i.e., $JD_p^{\perp} \subset T_p^{\perp}(M)$ for all $p \in M$.

The distributions D and $D^{\perp}$ on M are known as *holomorphic* and *totally real distributions* respectively.

If dim $D_p^{\perp}$ =dim $T_p^{\perp}M$, then the CR-submanifold is an *anti-holomorphic submanifold*. A CR-submanifold M is called *proper* if it is neither holomorphic nor totally real, i.e., both the distributions D and $D^{\perp}$ on M are non-trivial.

Definition 1.4.4. [15] A submanifold of an almost Hermitian manifold $\bar{M}$ is said to be a *CR-product* if it is locally a Riemannian product of a holomorphic submanifold M_T and a totally real submanifold $M_\perp$ of $\bar{M}$.

Obviously, a CR-product is a CR-submanifold whereas a CR-submanifold is a CR-product if and only if the distributions D and $D^\perp$ are involutive and their leaves are totally geodesic in M, or equivalently D and $D^\perp$ are parallel on M, i.e., $\nabla_X Y \in D$ and $\nabla_Z W \in D^\perp$ for each $X, Y \in D$ and $Z, W \in D^\perp$.

Remark 1.4.1. Notice that a distribution D on M is holomorphic if and only if for any non zero vector $U \in D$ at any point $p \in M$, the angle between D_p and JU is equal to zero whereas D is totally real if and only if for any non zero tangent vector $U \in D$ at any point $p \in M$, the angle between JU and D_p is equal to $\pi/2$. This point of view gives rise to the notion of slant distribution.

Definition 1.4.5. Given a submanifold M, isometrically immersed in an almost Hermitian manifold $(\bar{M}, J, g)$, a differentiable distribution D on M is said to be a *slant distribution* if for any non zero vector $U \in D_p$, $p \in M$, the angle between JU and the vector space D_p is constant, i.e., it is independent of the choice of $p \in M$ and $U \in D_p$. This constant angle is called the *wirtinger angle* of the slant distribution D. A submanifold M of $\bar{M}$ is called a *slant submanifold* if the tangent bundle TM is slant.

The notion of slant submanifolds provides a generalization of holomorphic and totally real submanifolds. In fact, for the wirtinger angle $\theta = 0$ and $\pi/2$, a slant submanifold is holomorphic and totally real respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real. If M is a slant submanifold of an almost Hermitian manifold $\bar{M}$, then we have (cf.[17])

$$P^2 = -cos^2(\theta)I, \tag{1.4.7}$$

where θ is the wirtinger angle of M in $\bar{M}$. Hence, we have

$$g(PU, PV) = cos^2(\theta)g(U, V), \tag{1.4.8}$$

$$g(FU, FV) = sin^2(\theta)g(U, V), \tag{1.4.9}$$

for U, V tangent to M.

Definition 1.4.6. [48] A submanifold M of an almost Hermitian manifold $(\bar{M}, J, g)$ is said to be a *semi-slant submanifold* if it is endowed with two orthogonal distributions D and D^θ, where D is invariant with respect to J and D^θ is slant with wirtinger angle θ, i.e., the angle θ between JX and D_p^θ is constant for each $X \in D_p^\theta$ and $p \in M$.

It is clear that CR-submanifolds and slant submanifolds are in particular semi-slant submanifolds with $\theta = \frac{\pi}{2}$ and $D = \{0\}$ respectively. A semi-slant submanifold is called *proper* if $\theta \neq \frac{\pi}{2}$.

On a semi-slant submanifold M of an almost Hermitian manifold $\bar{M}$, the tangent bundle TM and the normal bundle $T^\perp M$ are decomposed as

$$TM = D \oplus D^\theta \tag{1.4.10}$$

and

$$T^\perp M = FD^\theta \oplus \mu \tag{1.4.11}$$

where μ is the orthogonal complementary distribution to FD^θ in $T^\perp M$ and is invariant under J. That means $J\xi = f\xi$ for each $\xi \in \mu$ whereas $f\xi \in FD^\theta$ for each $\xi \in FD^\theta$. Moreover, following are some other easy observations

$$\begin{cases} (a)\ FD = \{0\}, & (b)\ PD = D, \\ (c)\ PD^\theta \subseteq D^\theta, & (d)\ t(T^\perp M) \subseteq D^\perp. \end{cases} \tag{1.4.12}$$

In terms of P, F, t and f, we have

$$\begin{cases} (e)\ P^2 + tF = -I, & (f)\ f^2 + Ft = -I, \\ (g)\ FP + fF = 0, & (h)\ tf + Pt = 0. \end{cases} \tag{1.4.13}$$

There is yet another genaralization of CR-submanifolds known as generic submanifolds. These submanifolds are defined by relaxing the condition on the complementary distribution of the holomorphic distribution.

Definition 1.4.7. [14]. Let M be a real submanifold of an almost Hermitian manifold $\bar{M}$. Then for each $p \in M$, let $D_p = T_pM \cap JT_pM$ be the maximal holomorphic subspace of $T_p(M)$. If $D : p \to D_p$ is a smooth holomorphic distribution on M, then M is called a *generic submanifold* of $\bar{M}$. The complementary distribution D^0 of D is called a *purely real distribution* on M. A generic submanifold is a CR-submanifold if the purely real distribution on M is totally real. A purely real distribution D^0 on a generic submanifold M is called *proper* if it is not totally real. A generic submanifold is called *proper* if the purely distribution is proper.

Definition 1.4.8. A generic submanifold M in a Kaehler manifold is a *purely real* (respectively, *complex) submanifold* if $D_p = \{0\}$ (respectively $D_p = T_pM$).

It is easy to observe that the relations (1.4.12) and (1.4.13) hold on a generic submanifold of an almost Hermitian manifold.

A generic submanifold M in a Kaehler manifold $\bar{M}$ is a *CR-submanifold* if the orthogonal complementary distribution $D^\perp$ of D in TM is totally real, i.e. $JD_p^\perp \subseteq T_p^\perp M$, $T_p^\perp M$ the normal space of M at p. A CR-submanifold is a *totally real submanifold* if $D_p = \{0\}$. A CR-submanifold is an *anti-holomorphic submanifold* if $JD_p^\perp = T_p^\perp M$, i.e., $(\mu = 0)$.

For a generic submanifold M in a Kaehler manifold $\bar{M}$, the orthogonal complementary distribution $D^\perp$, called *purely real distribution*, satisfies

$$D_p \perp D_p^\perp, \quad PD_p^\perp \subseteq D_p^\perp,$$

$$D_p^\perp \cap JD_p^\perp = \{0\}$$

Let ν_p be the vector space of *holomorphic normal vectors* to M at p, or simply the *holomorphic normal space* of M at p, i.e.,

$$\nu_p = T_p^\perp M \cap JT_p^\perp M$$

Then, ν_p defines a differentiable vector subbundle of $T^\perp M$. It is easy to verify that

$$T^\perp M = FD^\perp \oplus \nu, \quad t(T^\perp M) = D^\perp \quad \textit{and} \tag{1.4.14}$$

$$g(FD^\perp, \nu) = 0 \tag{1.4.15}$$

A vector subbundle μ of the normal bundle $T^{\perp}M$ is said to be *parallel* (in the normal bundle) if

$$\nabla^{\perp}_{X}\xi \in \mu \tag{1.4.16}$$

for any X in TM and any local cross-section ξ in μ.

CHAPTER 2

GENERIC SUBMANIFOLDS OF A KAEHLER MANIFOLD

To study the geometry of a Riemannian manifold, it is convenient to first embed it into a known manifold and then study its geometry vis-a-vis the known ambient manifold. If the ambient manifold happens to be an almost Hermitian manifold, we expect on the submanifold, invariant, anti-invariant and slant distributions with respect to the Hermitian structure on the ambient manifold. A. Bejancu [2] developed the notion of CR-submanifolds, thus providing a single setting to study invariant and anti-invariant submanifolds. Further generalized versions like semi-slant and generic submanifolds emerged at a later stage (cf. [48],[14]). A significant contribution in this area is made by B.Y.Chen [15],[16] etc. The present chapter deals with these studies.

2.1. Generic Submanifolds of an Almost Hermitian Manifold

To study the geometry of a generic submanifold, it is important to know the properties of the canonical projections P, F, t and f. The following lemma provides information about them.

Lemma 2.1.1. *Let M be generic submanifold of an almost Hermitian manifold $\bar{M}$ then ,*

$$PD \in D, \quad PD^{\perp} \in D^{\perp}, \quad tN \in D^{\perp}, \quad FU \in FD^{\perp}.$$

Moreover, if M is a CR-submanifold then $PU \in D$, $f\xi \in \mu$ for each $U \in D$ and $\xi \in T^{\perp}M$.

Further, let $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ denote respectively the tangential and normal parts of $(\bar{\nabla}_U J)V$, then by an easy computation, we obtain the following formulae

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{2.1.1}$$

$$(2.1.2) \qquad \mathcal{Q}_U V = (\bar{\nabla}_U F)V + h(U, PV) - fh(U,V).$$

Similarly, for $\xi \in T^{\perp}M$ denoting by $\mathcal{P}_U\xi$ and $\mathcal{Q}_U\xi$ respectively, the tangential and normal parts of $(\bar{\nabla}_U J)\xi$, we find that

$$(2.1.3) \qquad \mathcal{P}_U\xi = (\bar{\nabla}_U t)\xi + PA_\xi U - A_{f\xi}U,$$

$$(2.1.4) \qquad \mathcal{Q}_U\xi = (\bar{\nabla}_U f)\xi + h(t\xi, U) + FA_\xi U.$$

The following properties of $\mathcal{P}$and $\mathcal{Q}$ can be verified through straight forward computations.

p$_1$. (i) $\mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W$; (ii) $\mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W,$

p$_2$. (i) $\mathcal{P}_U(V+W) = \mathcal{P}_U V + \mathcal{P}_U W$; (ii) $\mathcal{Q}_U(V+W) = \mathcal{Q}_U V + \mathcal{Q}_U W,$

p$_3$. (i) $g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W)$; (ii) $g(\mathcal{Q}_U V, \xi) = -g(V, \mathcal{P}_U \xi),$

p$_4$. $\mathcal{P}_U JV + \mathcal{Q}_U JV = -J(\mathcal{P}_U V + \mathcal{Q}_U V),$

for all U, V and W in TM and ξ in $T^{\perp}$M.

On a submanifold M of a nearly Kaehler manifold $\bar{M}$, it follows from (1.3.12) that

$$(2.1.5) \qquad (a)\ \mathcal{P}_U V + \mathcal{P}_V U = 0, \qquad (b)\ \mathcal{Q}_U V + \mathcal{Q}_V U = 0$$

Throughout, we assume that M is a generic submanifold of an almost Hermitian manifold $\bar{M}$ with holomorphic distribution D and the purely real distribution $D^{\perp}$. It is known that if D is parallel then it is clearly integrable and its leaves are totally geodesic in M. If D is parallel then the orthogonal complementary distribution $D^{\perp}$ is also parallel which implies that D is parallel if and only if $D^{\perp}$ is parallel. In this case, M is locally the Riemannian product of the leaves of D and $D^{\perp}$. If M is a CR-submanifold with parallel holomorphic and totally real distributions, then M is a CR-product.

The above observations can be re-stated as the following lemma.

Lemma 2.1.2. [37] *Let M be a generic submanifold of an almost Hermitian manifold $\bar{M}$. Then M is locally the Riemannian product of the leaves of D and $D^{\perp}$ if and only if*

$$\nabla_U X \in D \quad or \quad \nabla_U Z \in D^{\perp}$$

for each $X \in D, Z \in D^{\perp}$ and $U \in TM$.

The following is an easy consequence of the above lemma and can be proved on taking account equation (1.4.3).

Corollary 2.1.1. [37] *If a generic submanifold of an almost Hermitian manifold $\bar{M}$ is a Riemannian product of the leaves of D and $D^{\perp}$, then*

$$(\bar{\nabla}_U P)X \in D \quad \textit{or equivalently} \quad (\bar{\nabla}_U P)Z \in D^{\perp}$$

for each $U \in TM$, $X \in D$ and $Z \in D^{\perp}$.

Remark 2.1.1. The converse of Corollary 2.1.1 is true when M is a CR-submanifold. In this case we have a stronger result, i.e., the following corollary.

Corollary 2.1.2. [37] *A CR-submanifold M in an almost Hermitian manifold $\bar{M}$ is a CR-product if and only if*

$$(\bar{\nabla}_U P)V \in D$$

for each $U, V \in TM$.

In terms of the normal valued 1-form F, we have the following characterization for M to be a Riemannian product in $\bar{M}$.

Corollary 2.1.3. [37] *A generic submanifold M of an almost Hermitian manifold $\bar{M}$ is a Riemannian product of the leaves of D and $D^{\perp}$ if and only if*

$$(\bar{\nabla}_U F)X = 0$$

for each $U \in TM$, $X \in D$.

The proof follows by putting $V = X$ in equation (1.4.4) and taking account of (1.4.12) (b).

In the following, we obtain integrability conditions for the distributions D and $D^{\perp}$ on a generic submanifold of an almost Hermitian manifold.

Proposition 2.1.1. [37] *The holomorphic distribution D on a generic submanifold of an almost Hermitian manifold is integrable if and only if*

$$\mathcal{Q}_X Y - \mathcal{Q}_Y X = h(X, PY) - h(Y, PX)$$

for each $X, Y \in D$.

Proof. For $\xi \in T^{\perp}M$, we have

$$g(\bar{\nabla}_X JY - \bar{\nabla}_Y JX, \xi) = g(h(X, PY) - h(PX, Y), \xi)$$

or

$$g(J(\bar{\nabla}_X Y - \bar{\nabla}_Y X) + \mathcal{Q}_X Y - \mathcal{Q}_Y X, \xi) = g(h(X, PY) - h(PX, Y), \xi)$$

or

$$g(F[X, Y], \xi) = g(h(X, PY) - h(PX, Y) + \mathcal{Q}_Y X - \mathcal{Q}_X Y, \xi).$$

The assertion follows from the above relation.

Proposition 2.1.2. [37] *The purely real distribution $D^{\perp}$ on a generic submanifold of an almost Hermitian manifold is integrable if and only if*

$$\nabla_Z PW - \nabla_W PZ + A_{FZ} W - A_{FW} Z + \mathcal{P}_W Z - \mathcal{P}_Z W$$

lies in $D^{\perp}$ for each $Z, W \in D^{\perp}$.

Proof. For $X \in D$, we find that

$$\begin{aligned} g(P[Z, W], X) &= g(\bar{\nabla}_Z JW - \bar{\nabla}_W JZ + \mathcal{P}_W Z - \mathcal{P}_Z W, X) \\ &= g(\nabla_Z PW - \nabla_W PZ + A_{FZ} W - A_{FW} Z + \mathcal{P}_W Z - \mathcal{P}_Z W, X) \end{aligned}$$

which proves the assertion.

2.2. Generic Submanifolds of a Kaehler Manifold

Lemma 2.2.1. [14] *Let M be a generic submanifold of a Kaehler manifold $\bar{M}$. Then*

$$g(h(JX, U), \xi) = g(Jh(X, U), \xi) \tag{2.2.1}$$

for any vector X in D, U in TM and ξ in μ.

The proof of the lemma follows from Gauss formula.

Proposition 2.2.1. [14] *Let M be a generic submanifold in a Kaehler manifold $\bar{M}$. Then the holomorphic distribution D is integrable if and only if*

$$g(h(X, JY), FZ) = g(h(JX, Y), FZ) \tag{2.2.2}$$

for any vector X, Y in D and Z in $D^{\perp}$.

Proof. Since $\bar{M}$ is Kaehlerian, Gauss formula gives

$$h(X, JY) - h(JX, Y) = J[X, Y] + \nabla_Y JX - \nabla_X JY \tag{2.2.3}$$

for any vector fields X, Y in D. If the holomorphic distribution D is integrable, the right hand side of (2.2.3) lies in TM, whereas the left hand side is a vector field normal to M, therefore both the sides will separately be zero. Thus we obtain $h(X, JY) = h(JX, Y)$. In particular, we have (2.2.2).

Conversely, if (2.2.2.) holds, then

$$h(X, JY) - h(JX, Y) \in \mu. \tag{2.2.4}$$

Now, for $\xi \in \mu$,

$$g(h(X, JY) - h(JX, Y), \xi) = g(h(X, JY), \xi) - g(h(JX, Y), \xi)$$

The right hand side of the above relation is zero by virtue of (2.2.1). Therefore

$$h(X, JY) - h(JX, Y) \in FD^{\perp} \tag{2.2.5}$$

From the observation (2.2.4) and (2.2.5), we have

$$h(X, JY) = h(JX, Y)$$

for any vectors X, Y in D. Thus by (2.2.3) we obtain $J[X, Y] = \nabla_Y JX - \nabla_X JY$. Since $\nabla_Y JX - \nabla_X JY$ is tangent to M, this implies that $[X, Y]$ lies in D. The proposition thus follows from the Frobenius theorm.

The proof of the proposition also follows from Proposition 2.1.1 as $\mathcal{Q}_U V = 0, \ \forall \ U, V \in TM$, in the setting of Kaehler manifolds.

Proposition 2.2.2. [14] *Let M be a generic submanifold in a Kaehler manifold $\bar{M}$. Then the purely real distribution $D^{\perp}$ is integrable if and only if*

$$\nabla_Z(PW) - \nabla_W(PZ) + A_{FZ}W - A_{FW}Z \ \textit{ lies in } \ D^{\perp} \tag{2.2.6}$$

for any vector fields Z, W in $D^{\perp}$.

Proof. For any vector fields Z, W in $D^{\perp}$, Gauss and Weingarten formulae and (1.4.1) give

$$\begin{aligned} J\nabla_Z W + Jh(Z, W) &= \bar{\nabla}_Z(PW) + \bar{\nabla}_Z(FW) \\ &= \nabla_Z(PW) + h(Z, PW) - A_{FW}Z + \nabla^{\perp}_Z(FW), \end{aligned}$$

from which we obtain

$$\nabla_Z W = PA_{FW}Z - P\nabla_Z(PW) - th(Z,PW) - t\nabla^{\perp}_Z(FW). \tag{2.2.7}$$

Thus we get

$$\begin{aligned}[Z,W] = {} & P\{A_{FW}Z - A_{FZ}W + \nabla_W(PZ) - \nabla_Z(PW)\} \\ & + t\{h(W,PZ) - h(Z,PW) + \nabla^{\perp}_W(FZ) - \nabla^{\perp}_Z(FW)\}.\end{aligned} \tag{2.2.8}$$

Since $t(T^{\perp}M) = D^{\perp}$, this implies that $[Z,W]$ lies in $D^{\perp}$ if and only if (2.2.6) holds. The proposition is proved.

As $\mathcal{P}_U V = 0$ on a submanifold of a Kaehler manifold, the above proposition follows from Proposition 2.1.2.

Lemma 2.2.2. [14] *Let M be a generic submanifold of a Kaehler manifold $\bar{M}$. If D is integrable and its leaves are totally geodesic in M, then*

$$g(h(D,D), FD^{\perp}) = 0$$

.

Proof. Under the hypothesis $\nabla_X Z$ lies in $D^{\perp}$ for any vector fields X in D and Z in $D^{\perp}$. So for any vector field Y in D we have

$$\begin{aligned} 0 &= g(\nabla_X Z, JY) \\ &= g(\bar{\nabla}_X Z, JY) \\ &= -g(\bar{\nabla}_X JZ, Y) \\ &= g(A_{FZ}X, Y) - g(\nabla_X PZ, Y) \\ &= g(A_{FZ}X, Y). \\ &= g(h(X,Y), FZ) \end{aligned}$$

This proves the lemma.

Remark 2.2.1. Although the converse of Lemma 2.2.2 is true for CR-submanifolds, it is not true in general if M is just a generic submanifold.

Lemma 2.2.3. [14] *Let M be a generic submanifold of a Kaehler manifold $\bar{M}$. If $D^{\perp}$ is integrable and its leaves are totally geodesic in M,then $g(h(D,D^{\perp}), FD^{\perp}) = 0$.*

Proof. Under the hypothesis, for any X in D and Z, W in $D^{\perp}$, we have

$$\begin{aligned}0 &= g(\nabla_Z X, W) = g(\bar{\nabla}_Z JX, JW) \\ &= g(\bar{\nabla}_Z JX, PW) + g(\bar{\nabla}_Z JX, FW) \\ &= g(h(JX, Z), FW)\end{aligned}$$

From this we obtain the lemma.

If the purely real distribution $D^{\perp}$ in a generic submanifold is slant with wirtinger angle θ, i.e., $D^{\perp} = D^{\theta}$, then the generic submanifold reduces to a semi-slant submanifold.

On a semi-slant submanifold M of an almost Hermitian manifold, in view of the decomposition (1.4.11), we write

$$h(U, V) = h_{FD^{\theta}}(U, V) + h_{\mu}(U, V) \tag{2.2.9}$$

where $h_{FD^{\theta}}(U, V)$ and $h_{\mu}(U, V)$ are the components of $h(U, V)$ in FD^{θ} and μ respectively. Moreover, if $\{Z_1, Z_2, ...Z_q\}$ be a local orthonormal frame of vector fields on D^{θ}, then

$$h_{FD^{\theta}}(U, V) = \sum_{r=1}^{q} h^r(U, V) F Z_r \tag{2.2.10}$$

where

$$h^r(U, V) = \csc^2\theta g(h(U, V), FZ_r) \tag{2.2.11}$$

Therefore,

$$\begin{aligned}&g(h_{FD^{\theta}}(U_1, V_1), h_{FD^{\theta}}(U_2, V_2)) \\ &\qquad = \csc^2\theta \sum_r g(h(U_1, V_1), FZ_r) g(h(U_2, V_2), FZ_r),\end{aligned} \tag{2.2.12}$$

for any $U_1, U_2, V_1, V_2 \in TM$.

Theorem 2.2.1. [39] *On a semi-slant submanifold M of an almost Hermitian manifold,*

$$\|h_{FD^{\theta}}(U, V)\|^2 = \csc^2\theta \sum_r g(h(U, V), FZ_r)^2, \tag{2.2.13}$$

$$\|f h_{FD^{\theta}}(U, V)\|^2 = \cot^2\theta \sum_r g(h(U, V), FZ_r)^2 \tag{2.2.14}$$

for any $U, V \in TM$.

Proof. Formula (2.2.13) is an immediate consequence of (2.2.12). On the other hand, it is easy to deduce from (1.4.13)(g) that

$$g(fh(U,V), FZ) = g(h(U,V), FPZ) \tag{2.2.15}$$

Now, (2.2.14) is obtained on using (2.2.10), (2.2.15), (1.4.8) and (1.4.9). □

In particular, if M is a CR-submanifold of a Kaehler manifold $\bar{M}$, then on using Gauss and Weingarten formulae, we have

$$J\nabla_U Z + Jh(U,Z) = -A_{JZ}U + \nabla_U^{\perp} JZ \tag{2.2.16}$$

for U tangent to M and Z in $D^{\perp}$.

Lemma 2.2.4. [15] *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$.*
Then

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X), \tag{2.2.17}$$

$$A_{JZ}W = A_{JW}Z, \tag{2.2.18}$$

$$A_{J\xi}X = -A_{\xi}JX \tag{2.2.19}$$

for U tangent to M, X in D, Z and W in $D^{\perp}$, and ξ in μ.

Proof. Taking the inner product of (2.2.16) with JX,

$$g(J\nabla_U Z, JX) = g(-A_{JZ}U, JX)$$

Hence (2.2.17) follows.
For (2.2.18), we consider $g(A_{JZ}W, U)$

$$\begin{aligned} g(A_{JZ}W, U) &= g(h(U,W), JZ) \\ &= -g(J(\bar{\nabla}_U W - \nabla_U W), Z) \\ &= -g(\bar{\nabla}_U JW, Z) + g(J\nabla_U W, Z) \\ &= g(A_{JW}U, Z) - g(\nabla_U^{\perp} JW, Z) - g(\nabla_U W, JZ). \end{aligned}$$

Thus we get
$$g(A_{JZ}W, U) = g(A_{JW}Z, U),$$
for all $U \in TM$. This verifies equation (2.2.18).

Formula (2.2.19) follows from the fact that
$$g(h(JX, Y), \xi) = g(\bar{\nabla}_Y JX, \xi) = (Jh(X, Y), \xi).$$

Lemma 2.2.5. [15] *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then for any Z, W in $D^{\perp}$ we have*

$$\nabla^{\perp}_W JZ - \nabla^{\perp}_Z JW \in JD^{\perp}. \tag{2.2.20}$$

Proof. For any ξ in μ and Z, W in $D^{\perp}$, we have
$$g(A_{J\xi}Z, W) = -g(\bar{\nabla}_Z J\xi, W) = g(\nabla^{\perp}_Z \xi, JW) = -g(\xi, \nabla^{\perp}_Z JW).$$
Thus we obtain
$$g(\xi, \nabla^{\perp}_W JZ - \nabla^{\perp}_Z JW) = g(A_{J\xi}Z, W) - g(A_{J\xi}W, Z) = 0.$$
Since this is true for all ξ in μ, (2.2.20) holds.

From Lemma 2.2.4, it follows that we have $J[Z, W] = J(\nabla_Z W - \nabla_W Z) = \nabla^{\perp}_Z JW - \nabla^{\perp}_W JW$. Thus using Lemma 2.2.5, $[Z, W] \in D^{\perp}$. Thus we obtain

Lemma 2.2.6. [15] *The totally real distribution $D^{\perp}$ on a CR-submanifold in a Kaehler manifold is integrable.*

This theorem has been generalized to CR-submanifolds in a locally conformal almost Kaehler manifolds in [8].

For the holomorphic distribution D on a CR-submanifold of a Kaehler manifold, the integrability conditions (2.2.2) remain the same, i.e., we have,

Lemma 2.2.7. [15] *Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then the holomorphic distribution D is integrable if and only if*

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ), \tag{2.2.21}$$

for any X, Y in D and Z in $D^{\perp}$.

Lemma 2.2.8. [15] *The leaves of the holomorphic distribution D on a CR-submanifold M of a Kaehler manifold $\bar{M}$ are totally geodesic in M if and only if*

$$g(A_{JD^\perp}D, D) = 0. \tag{2.2.22}$$

Proof. For X, Y in D, Z in $D^\perp$,

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) \\ &= g(\bar{\nabla}_X JY, JZ) \end{aligned}$$

Therefore,

$$g(\nabla_X Y, Z) = g(h(X, JY), JZ). \tag{2.2.23}$$

If (2.2.22) is satisfied then by the Lemma 2.2.7, D is integrable. Further, from (2.2.23) its leaves are totally geodesic in M.

Conversely, if the leaves of D are totally geodesic in M, then in view of (2.2.23), (2.2.22) holds.

Lemma 2.2.9. [15] *On a CR-submanifold M in a Kaehler manifold $\bar{M}$, leaves of $D^\perp$ are totally geodesic in M if and only if*

$$g(A_{JD^\perp}D, D^\perp) = 0. \tag{2.2.24}$$

Proof. If the leaves of $D^\perp$ are totally geodesic in M, then by definition

$$\nabla_Z W \in D^\perp,$$

for all $Z, W \in D^\perp$. That is,

$$g(\nabla_Z W, X) = 0,$$

for all $X \in D$. Therefore

$$\begin{aligned} 0 = g(\nabla_Z W, X) = g(W, \nabla_Z X) = g(\bar{\nabla}_Z X, W) \\ = g(\bar{\nabla}_Z JX, JW) = g(h(JX, Z), JW) \end{aligned}$$

Hence, the leaves of $D^\perp$ are totally geodesic if and only if

$$g(h(D, D^\perp), JD^\perp) = 0$$

Lemma 2.2.10. [15] *If* (2.2.24) *holds and D is integrable, then for any X in D and ξ in $JD^{\perp}$, we have*

$$A_{\xi}JX = -JA_{\xi}X \tag{2.2.25}$$

If the two distributions on a CR-submanifold M of a Kaehler manifold are involutive and their leaves are totally geodesic in M, then M is a CR-product. B.Y.Chen [15] obtained characterization under which a CR-submanifold reduces to a CR-product:

Theorem 2.2.2. [15] *A CR-submanifold M of a Kaehler manifold $\bar{M}$ is a CR-product if and only if P is parallel, i.e., $\bar{\nabla}P = 0$.*

Proof. As $\bar{M}$ is Kaehler,

$$\bar{\nabla}_U JV = J\bar{\nabla}_U V,$$

for any vector fields U, V tangent to M. On using (1.4.1) and Gauss formula, the above equation becomes,

$$\bar{\nabla}_U(PV + FV) = J(\nabla_U V + h(U,V)),$$

which on applying Gauss-Weingarten formulae gives

$$\nabla_U PV + h(U,PV) - A_{FV}U + \nabla_U^{\perp}FV = P\nabla_U V + F\nabla_U V + th(U,V) + fh(U,V).$$

Now comparing tangential parts in both sides of the above equation and using (1.4.3) we get

$$(\bar{\nabla}_U P)V = th(U,V) + A_{FV}U.$$

If P is parallel then from the above equation

$$th(U,V) = -A_{FV}U,$$

for any vector fields U, V tangent to M. In particular if X in D then $FX = 0$. Hence above equation implies

$$th(U,X) = 0,$$

$$\text{i.e.,}\ \ A_{JZ}X = 0,$$

for any Z in $D^{\perp}$ and X in D. Thus by Lemma 2.2.7 and 2.2.8, D is integrable and its leaves are totally geodesic in M. Similarly on using Lemma 2.2.9, leaves of $D^\perp$ are totally geodesic in M. Thus M is a CR-product.

Conversely, if M is a CR-product, then

$$g(\nabla_X Y, Z) = 0 \qquad \text{and} \qquad g(\nabla_Z W, Y) = 0,$$

for all X, Y in D and Z, W in $D^\perp$. That is, we have

$$\nabla_X Y \in D \qquad \text{and} \quad \nabla_Z Y \in D,$$

$$\text{i.e.,} \qquad \nabla_U Y \in D,$$

for all U in TM, Y in D. On using the fact that M is a CR-product and Gauss formula we obtain

$$Jh(U,Y) = h(U,JY). \tag{2.2.26}$$

We have

$$(\bar{\nabla}_U P)Y = th(U,Y) + A_{FY}U, .$$

As $FY = 0$ for all $Y \in D$ and $th(U,Y) = 0$ for all $U \in TM$ and $Y \in D$, from (2.2.26) we have

$$(\bar{\nabla}_U P)V = 0. \tag{2.2.27}$$

Similarly, as $\nabla_U Z \in D^\perp$, for any Z in $D^\perp$ and U tangent to M, it is easy to see that

$$(\bar{\nabla}_U P)Z = 0. \tag{2.2.28}$$

Hence, if M is CR-product, then by (2.2.27) and (2.2.28),

$$\bar{\nabla}_U P = 0.$$

This proves the theorem completely.

From the proof of Theorem 2.2.2 we have the folllowing.

Lemma 2.2.11. [15] *A CR-submanifold M in a Kaehler manifold $\bar{M}$ is a CR-product if and only if $A_{JD^\perp}D = 0$.*

Remark 2.2.2. In [4] Bejancu-Kon-Yano proved that if M is an anti-holomorphic submanifold and $\bar{\nabla}P = 0$, then M is a CR-product.

CHAPTER 3

WARPED PRODUCT SUBMANIFOLDS IN A KAEHLER MANIFOLD

A CR-submanifold M of a Kaehler manifold $\bar{M}$ is called CR-product if it is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$ of $\bar{M}$. The notion of CR-products in Kaehler manifolds was introduced in [15] and it was proved that a CR-submanifold M in a complex Euclidean space is a CR-product if and only if it is a Riemannian product of a holomorphic submanifold of a linear complex subspace and a totally real submanifold of a linear complex subspace. However, there do not exist CR-products in complex hyperbolic spaces other than holomorphic and totally real submanifolds. As warped products are generalized version of Riemannian product manifolds, it is natural to seek existence of warped product manifolds in Kaehler as well as in non-Kaehler manifolds. B.Y.Chen [18] initiated the investigations with this stand point and explored CR-submanifolds as warped product manifolds in Kaehler manifolds. Subsequently, B.Sahin [49] studied semi-slant warped products in a Kaehler manifold. The findings of Chen and Sahin are discussed in the present chapter.

3.1. CR-Submanifolds as Warped Product Submanifolds in Kaehler Manifolds

B.Y.Chen [18], [19] initiated the study of warped product manifolds with extrinsic geometric point of view when he considered CR-submanifolds of a Kaehler manifold embedded as warped product manifolds. That is how, the notion of warped product submanifolds emerged. The definition can be formulated as:

Let $(\bar{M}, g)$ be a Riemannian manifold and M a submanifold of $\bar{M}$. Then M is called a warped product submanifold of $\bar{M}$ if it satisfies

(i) M is a Riemannian submanifold of $\bar{M}$.

(ii) M is a warped product manifold of two submanifolds M_1 and M_2 of $\bar{M}$.

(*iii*) The two submanifolds are orthogonal, i.e., $g(U_1, U_2) = 0$, for any $U_1 \in TM_1$ and $U_2 \in TM_2$.

In this section we study CR-submanifolds in a Kaehler manifold $\bar{M}$ which are warped products of the form $N_\perp \times_f N_T$, where $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$.

Theorem 3.1.1. [18] *If $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of a Kaehler manifold $\bar{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$, then M is a CR-product.*

Proof. Let $M = N_\perp \times_f N_T$ be a warped product CR-submanifold in a Kaehler manifold $\bar{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$. Since the metric tensor of M is given by $g = g_{N_\perp} + f^2 g_{N_T}$, $N_\perp$ is a totally geodesic submanifold of M. Thus, for any vector fields Z, W on $N_\perp$ and X on N_T, we have

$$g(\nabla_Z W, X) = 0. \tag{3.1.1}$$

Further, as $\bar{M}$ is Kaehlerian, the formulae of Gauss and Weingarten imply that

$$-A_{JW}Z + \nabla_Z^\perp(JW) = J(\nabla_Z W) + Jh(Z, W). \tag{3.1.2}$$

Taking the inner product of (3.1.2) with JX gives

$$g(A_{JW}Z, JX) = -g(\nabla_Z W, X). \tag{3.1.3}$$

By combining (3.1.1) and (3.1.3) we obtain

$$g(h(D, D^\perp), JD^\perp) = 0. \tag{3.1.4}$$

On the other hand, from Proposition 1.2.1, we have

$$\nabla_X Z = \nabla_Z X = (Z\ lnf)X, \tag{3.1.5}$$

for any vector fields X in D and Z in $D^\perp$. Thus, if we denote by h^T and A^T the second fundamental form and the shape operator of N_T in M, then we obtain from the formulae of Gauss and Weingarten that

$$g(h^T(X, Y), Z) = g(A_Z^T X, Y) = -g(\nabla_X Z, Y) = -(Z\ lnf)g(X, Y) \tag{3.1.6}$$

for any X, Y in D and Z in $D^\perp$. Hence, we find

$$h^T(X, Y) = -\nabla(lnf)g(X, Y), \tag{3.1.7}$$

where $\nabla(lnf)$ is the gradient of lnf. Equation (3.1.7) implies that N_T is a totally umbilical submanifold of M.

Let $\hat{h}$ denote the second fundamental form of N_T in the ambient space $\bar{M}$. Then

$$\hat{h}(X,Y) = h^T(X,Y) + h(X,Y), \tag{3.1.8}$$

for any X, Y tangent to N_T. By applying (3.1.7) and (3.1.8) we find

$$g(\hat{h}(X,X),Z) = -Z(lnf)g(X,X). \tag{3.1.9}$$

Since N_T is a holomorphic submanifold of $\bar{M}$, we also have the following relations:

$$\hat{h}(X,JY) = \hat{h}(JX,Y) = J\hat{h}(X,Y). \tag{3.1.10}$$

Hence by combining (3.1.9) and (3.1.10), we obtain

$$g(\hat{h}(X,X),Z) = -g(\hat{h}(JX,JX),Z) = Z(lnf)g(X,X) \tag{3.1.11}$$

(3.1.9) and (3.1.11) imply $Z(lnf) = 0$. Therefore, we obtain from (3.1.6) and (3.1.8) that

$$g(\hat{h}(X,Y),Z) = g(h^T(X,Y),Z) = 0 \tag{3.1.12}$$

for any X, Y in D and Z in $D^\perp$.

Hence, by (3.1.8),(3.1.10) and (3.1.12), we obtain

$$g(h(X,Y),JZ) = g(\hat{h}(X,Y),JZ) = -g(\hat{h}(X,JY),Z) = 0. \tag{3.1.13}$$

Therefore

$$g(h(D,D),JD^\perp) = 0. \tag{3.1.14}$$

Conditions (3.1.4) and (3.1.14) imply $A_{JD^\perp}D = 0$. Therefore, by applying Lemma 2.2.11, we conclude that $M = N_\perp \times_f N_T$ is a CR-product.

Theorem 3.1.1 shows that there do not exist warped product CR-submanifolds, i.e., warped products of the form $N_\perp \times_f N_T$ other than CR-products such that N_T is a holomorphic submanifold and $N_\perp$ is a totally real submanifold of $\bar{M}$.

Now, we consider warped product submanifolds by reversing the two factors N_T and $N_\perp$ i.e., warped prodct of the form $N_T \times_f N_\perp$. Such warped products are called *CR-warped product submanifolds.* Non-trivial CR-warped products do exist in Kaehler manifolds.

Here, we give an example of such CR-warped products. We denote by $\mathbb{R}^{2m}, m \geq 1$, the Euclidean $2m$ space with the standard metric. Then the canonical complex structure of $\mathbb{R}^{2m}$ is defined by

$$J(x_1, y_1, \ldots\ldots, x_m, y_m) = (-y_1, x_1, \ldots\ldots, -y_m, x_m). \tag{3.1.15}$$

Example 3.1.1. Consider in $\mathbb{R}^8$ the submanifold M given by equations

$$x_1 = t\ cos\theta,\ \ x_2 = s\ cos\theta,\ \ x_3 = t\ cos\varphi,\ \ x_4 = s\ cos\varphi$$
$$x_5 = t\ sin\theta,\ \ x_6 = s\ sin\theta,\ \ x_7 = t\ sin\varphi,\ \ x_8 = s\ sin\varphi,\ \ \theta, \varphi \in (0, \frac{\pi}{2}).$$

Then TM is spanned by $Z_t, Z_s, Z_\theta, Z_\varphi$, where

$$\begin{aligned}
Z_t &= cos\ \theta\ \partial x_1 + cos\ \varphi\ \partial x_3 + sin\ \theta\ \partial x_5 + sin\ \varphi\ \partial x_7,\\
Z_s &= cos\ \theta\ \partial x_2 + cos\ \varphi\ \partial x_4 + sin\ \theta\ \partial x_6 + sin\ \varphi\ \partial x_8,\\
Z_\theta &= -t\ sin\ \theta\ \partial x_1 - s\ sin\ \theta\ \partial x_2 + t\ cos\ \theta\ \partial x_5 + s\ cos\ \theta\ \partial x_6,\\
Z_\varphi &= -t\ sin\ \varphi\ \partial x_3 - s\ sin\ \varphi\ \partial x_4 + t\ cos\ \varphi\ \partial x_7 + s\ cos\ \varphi\ \partial x_8,
\end{aligned}$$

Using (3.1.15), we obtain that $D = span\{Z_t, Z_s\}$ is invariant with respect to J. Moreover, JZ_θ and JZ_φ are orthogonal to TM. Hence, $D^\perp = span\{Z_\theta, Z_\varphi\}$ is anti-invariant with respect to J. Thus M is a CR-submanifold of $\mathbb{R}^8$. Furthermore, we can derive that D and $D^\perp$ are integrable. Denoting the integral manifolds of D and $D^\perp$ by M_T and $M_\perp$, respectively, then the induced metric tensor is

$$\begin{aligned}
g &= 2dt^2 + 2ds^2 + (t^2 + s^2)(d\theta^2 + d\varphi^2)\\
&= g_{M_T} + (t^2 + s^2)g_{M_\perp}
\end{aligned}$$

Thus M is a CR-warped product submanifold of $\mathbb{R}^8$ with warping function $f = \sqrt{t^2 + s^2}$.

For CR-warped products in Kaehler manifolds, we have the following.

Lemma 3.1.1. [18] *Let $M = N_T \times_f N_\perp$ be a CR-warped product in a Kaehler manifold $\bar{M}$. Then for any X, Y in D and Z, W in $D^\perp$ we have* *(i)* $g(h(D,D),JD^{\perp})=0$;

(ii) $g(h(JX,Z),JW)=X(lnf)g(Z,W)$;

(iii) $\nabla^{\perp}_X JZ = J\nabla_X Z$, *whenever* $h(D,D^{\perp})\subset JD^{\perp}$;

(iv) $g(h(D,D^{\perp}),JD^{\perp})=0$ *if and only if* M *is a trivial warped product in* $\bar{M}$*, i.e., a CR-product.*

Proof. Since $\bar{M}$ is Kaehlerian, we have

$$\bar{\nabla}_X JZ = J\bar{\nabla}_X Z,$$

$$-A_{JZ}X+\nabla^{\perp}_X JZ = J\nabla_X Z + Jh(X,Z), \tag{3.1.16}$$

for any vector fields X, Y on N_T and Z on $N_{\perp}$. Thus, by taking the inner product of (3.1.16) with JY, we find

$$g(-A_{JZ}X,JY)+g(\nabla^{\perp}_X JZ,JY)=g(J\nabla_X Z,JY)+g(Jh(X,Z),JY),$$

$$\text{i.e.,}\qquad -g(A_{JZ}X,JY)=g(\nabla_X Z,Y),$$

or

$$g(h(X,JY),JZ)=-g(\nabla_X Z,Y). \tag{3.1.17}$$

On the other hand, Since $M = N_T\times_f N_{\perp}$ is a warped product, N_T is a totally geodesic submanifold of M. Thus, we also have $g(\nabla_X Z,Y)=0$. Combining this with (3.1.17), we get

$$g(h(D,D),JD^{\perp})=0.$$

This prove statement (i).

Now, if we denote by h and A the second fundamental form and the shape operator of the immersion of M in $\bar{M}$. Then we obtain from the formulae of Gauss and Weingarten that

$$g(h(JX,Z),JW)=-g(JA_{JW}Z,X). \tag{3.1.18}$$

On using Proposition 1.2.1, we get

$$g(h(JX,Z),JW)=-g(\nabla_Z W,X)$$

$$= g(\nabla_Z X, W)$$
$$= X(\ln f)g(Z, W).$$

for any X in TN_T and Z,W in $TN_\perp$. This proves statement (ii).

Since N_T is a totally geodesic submanifold in M, $\nabla_X Z \in D^\perp$. Thus $J\nabla_X Z \in JD^\perp$. On the other hand, condition $h(D, D^\perp) \subset JD^\perp$ implies $Jh(X, Z) \in TM$. Therefore by aplying (3.1.16), we obtain statement (iii).

If $g(h(D, D^\perp), JD^\perp) = 0$, then by statement (ii)

$$(X \ln f) = 0.$$

That means f is constant along N_T. Hence, $M = N_T \times_f N_\perp$ is CR-product.

Conversely, if M is a CR-product submanifold then by Lemma 2.2.11, $A_{JD^\perp} D = 0$. In particular,

$$g(h(D, D^\perp), JD^\perp) = 0.$$

This implies statement (iv).

Now we give the following characterization for a CR-submanifold to become a CR-warped product submanifold of a Kaehler manifold.

Theorem 3.1.2. [18] *A proper CR-submanifold M of a Kaehler manifold $\bar{M}$ is locally a CR-warped product if and only if*

$$A_{JZ}X = ((JX)\lambda)Z \tag{3.1.19}$$

for $X \in D$ and $Z \in D^\perp$ and for some function λ on M satisfying $W\lambda = 0$ where $W \in D^\perp$.

Proof. If M is a CR-warped product $N_T \times_f N_\perp$ in a Kaehler manifold $\bar{M}$, then statement (i) of Lemma 3.1.1 and Proposition 1.2.1 imply that $A_{JZ}X = -((JX)\ln f)Z$ for each $X \in D$ and $Z \in D^\perp$. Since f is a function on N_T, we also have $W(\ln f) = 0$ for all $W \in D^\perp$.

Conversely, assume that M is a proper CR-submanifold of a Kaehler manifold $\bar{M}$ satisfying

$$A_{JZ}X = ((JX)\lambda)Z,$$

for $X \in D$, $Z \in D^{\perp}$ and for some function λ on M, with $Z\lambda = 0$. Thus we have

$$g(h(D,D), JD^{\perp}) = 0 \tag{3.1.20}$$

It follows from (3.1.20) that the holomorphic distribution D is integrable and its leaves are totally geodesic in M. Also

$$\begin{aligned} g(((J^2X)\lambda)Z, W) &= g(((-X)\lambda)Z, W) \\ &= g(A_{JZ}JX, W) \\ &= -g(\overline{\nabla}_{JX}JZ, W) \\ &= g(J\overline{\nabla}_{JX}Z, W). \end{aligned}$$

Therefore

$$-X(\lambda)g(Z,W) = g(\overline{\nabla}_{JX}Z, JW)$$

$$= g(h(JX,Z), JW). \tag{3.1.21}$$

On the other hand, from Lemma 2.1.4 and (3.1.21) we have

$$\begin{aligned} g(\nabla_Z X, W) &= -g(\nabla_Z W, X) \\ &= -g(JA_{JW}Z, X) \\ &= g(h(JX,Z), JW) \\ &= -X(\lambda)g(Z,W). \end{aligned} \tag{3.1.22}$$

for X in D and Z, W in $D^{\perp}$. Since the totally real distribution $D^{\perp}$ of a CR-submanifold of a Kaehler manifold is always integrable (cf. Lemma 2.2.6), by equation (3.1.22) and the condition $W\lambda = 0, W \in D^{\perp}$, imply that each leaf of $D^{\perp}$ is an extrinsic sphere in M, i.e., a totally umbilical submanifold with parallel mean curvature vector. Hence, by Theorem 1.2.2, M is locally isometric to a warped product $N_T \times_f N_{\perp}$ of a holomorphic submanifold N_T and a totally real submanifold $N_{\perp}$ of M, where N_T is a leaf of D and $N_{\perp}$ is a leaf of $D^{\perp}$ and f is a certain warping function.

3.2. A General Inequality for CR-Warped Product Submanifolds

Chen [18] established an inequality for the squared norm of the second fundamental form of a CR-warped product submanifold of a Kaehler manifold.

This section is devoted to discuss the same.

Definition 3.2.1. For a real hyperspace M of a Kaehler manifod $\bar{M}$ with a unit normal vector field ξ, the tangent vector field $J\xi$ on M is called a characteristic vector field of M.

Definition 3.2.2. A unit tangent vector V on M is called a principal vector if V is an eigenvector of the shape operator A_ξ, the corresponding eigenvalue is called the principal curvature at V.

Let us denote the tangent bundles on N_T and $N_\perp$ by D and $D^\perp$ respectively and let $B_1 = \{X_1, X_2, ..., X_p, X_{p+1} = JX_1, ..., X_{2p} = JX_p\}$ and $B_2 = \{Z_1, Z_2, ..., Z_q\}$ be local orthonormal frame of vector fields on N_T and $N_\perp$ respectively with $2p$ and q being their real dimensions. Then by (1.2.7)

$$||h||^2 = \sum_{i,j=1}^{2p} g(h(X_i, X_j), h(X_i, X_j)) \tag{3.2.1}$$

$$+\sum_{i,=1}^{2p}\sum_{r=1}^{q} g(h(X_i, Z_r), h(X_i, Z_r)) + \sum_{r,s=1}^{q} g(h(Z_r, Z_s), h(Z_r, Z_s))$$

We have the following result for CR-warped products in Kaehler manifolds.

Theorem 3.2.1. [18] *Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold in a Kaehler manifold $\bar{M}$. We have*

(i) *The squared norm of the second fundamental form of M satisfies*

$$||h||^2 \geq 2p||\nabla(\ln f)||^2, \tag{3.2.2}$$

where p is the dimension of $N_\perp$.

(ii) *If the equality sign of (3.2.2) holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold in $\bar{M}$.*

(iii) *When M is anti-holomorphic and $p > 1$, the equality sign of (3.2.2) holds identically if and only if $N_\perp$ is a totally umbilical submanifold of $\bar{M}$.*

(iv) *If M is anti-holomorphic and $p = 1$, then the equality sign of (3.2.2) holds identically if and only if the characteristic vector field $J\xi$ of M is a principal vector field with zero as its principal curvature. (Note that* *in this case, M is a real hypersurface in $\bar{M}$.) Also, in this case, the equality sign in* (3.2.2) *holds identically if and only if M is a minimal hypersurface in $\bar{M}$.*

Proof. In view of the decomposition (1.4.11)

$$h(U,V) = h_{JD^\perp}(U,V) + h_\mu(U,V) \tag{3.2.3}$$

for each $U,V \in TM$, where $h_{JD^\perp}(U,V) \in JD^\perp$ and $h_\mu(U,V) \in \mu$ with

$$h_{JD^\perp}(U,V) = \sum_{r=1}^{q} h^r(U,V)JZ_r \quad \text{and} \quad h_\mu(U,V) = \sum_{\alpha=1}^{k} h^\alpha(U,V)\xi_\alpha$$

where $h^r(U,V)$ and $h^\alpha(U,V)$ are the components of $h(U,V)$ in $JD^\perp$ and μ respectively. That is

$$\begin{cases} h^r(U,V) = g(h(U,V), JZ_r), \\ h^\alpha(U,V) = g(h(U,V), \xi_\alpha) \end{cases} \tag{3.2.4}$$

where $\{\xi_1, \xi_2, ..., \xi_k\}$ is the frame of orthonormal vector fields of μ with k being its dimension. Now, for any $i \in \{1, 2, ..., 2p\}$ and $r \in \{1, 2, ..., q\}$,

$$\begin{aligned} g(h_{JD^\perp}(X_i, Z_r), h_{JD^\perp}(X_i, Z_r)) &= g(\sum_{s=1}^{q} h^s(X_i, Z_r)JZ_s, \sum_{s=1}^{q} h^s(X_i, Z_r)JZ_s) \\ &= \sum_{s=1}^{q} h^s(X_i, Z_r)g(h(X_i, Z_r)JZ_s) \end{aligned}$$

As, $g(h(Z_r, X_i)JZ_s) = 0$, for $r \neq s$ by Lemma 3.1.1 (ii), the above equation yields

$$g(h_{JD^\perp}(X_i, Z_r), h_{JD^\perp}(X_i, Z_r)) = h^r(X_i, Z_r)g(h(X_i, Z_r)JZ_r)$$

for each r. Thus, we have

$$g(h_{JD^\perp}(X_i, Z_r), h_{JD^\perp}(X_i, Z_r)) = (g(h(X_i, Z_r)JZ_r))^2 \tag{3.2.5}$$

By using Lemma 3.1.1 (ii), equation (3.2.5) becomes

$$g(h_{JD^\perp}(X_i, Z_r), h_{JD^\perp}(X_i, Z_r)) = (X_i lnf)^2$$

Therefore,

$$\sum_{i=1}^{2p}\sum_{r=1}^{q}\|h_{JD^{\perp}}(X_i,Z_r)\|^2=\sum_{i=1}^{2p}\sum_{r=1}^{q}(X_i lnf)^2$$

$$=2q\sum_{i=1}^{2p}g(\nabla lnf,X_i)^2 \tag{3.2.6}$$

Hence, on taking account of (3.2.1) and (3.2.3) in the above equation, we obtain

$$\|h\|^2\geq 2q\|\nabla(lnf)\|^2$$

For any vector fields X in D and Z, W in $D^{\perp}$, Lemma 2.2.4 and (1.2.5) imply that

$$g(\nabla_W Z,X)=g(JA_{JZ}W,X)=-g(h(JX,W),JZ). \tag{3.2.7}$$

Hence by using statement (*ii*) of Proposition 1.2.1 and (3.2.7), we find

$$g(\nabla_W Z,X)=-(X\ln f)g(Z,W). \tag{3.2.8}$$

On the other hand, if we denote by $h^{\perp}$, the second fundamental form of $N_{\perp}$ in
$M=N_{\top}\times_f N_{\perp}$, we get

$$g(h^{\perp}(Z,W),X)=g(\nabla_W Z,X)\quad . \tag{3.2.9}$$

Combining (3.2.8) and (3.2.9) we obtain,

$$h^{\perp}(Z,W)=-g(Z,W)\nabla(\ln f). \tag{3.2.10}$$

Now, assume that the equality case of (3.2.2) holds. From statement (*ii*) of Lemma 3.1.1, we have

$$g(h(Z,JX),JZ)=X(lnf). \tag{3.2.11}$$

Then we obtain from (3.2.11) that

$$h(D,D)=0,\qquad h(D^{\perp},D^{\perp})=0,\qquad h(D,D^{\perp})\subset JD^{\perp}. \tag{3.2.12}$$

Since $N_{\top}$ is a totally geodesic submanifold in M, the first condition in (3.2.12) implies that $N_{\top}$ is totally geodesic in $\bar{M}$.

On the other hand, (3.2.10) shows that $N_\perp$ is totally umbilical in M. Now the second condition in (3.2.12) implies that $N_\perp$ is also totally umbilical in $\bar{M}$. Moreover, from (3.2.12), we know that M is minimal in $\bar{M}$.

Let us assume that M is an anti-holomorphic CR-warped product in $\bar{M}$. Then, from statement (i) of Lemma 3.1.1, we get

$$h(D, D) = 0. \tag{3.2.13}$$

If $N_\perp$ is totally umbilical in $\bar{M}$, then there exists a normal vector field $\bar{H}$ of $N_\perp$ in $\bar{M}$ such that the second fundamental form $\bar{h}$ of $N_\perp$ in $\bar{M}$ satisfies

$$\bar{h}(Z, W) = g(Z, W)\bar{H}, \tag{3.2.14}$$

for Z, W tangent to $N_\perp$. Since

$$\bar{h}(Z, W) = h^\perp(Z, W) + h(Z, W),$$

(3.2.14) implies that there is a normal vector field η such that

$$h(Z, W) = g(Z, W)\eta, \tag{3.2.15}$$

Hence, for each unit vector $W \in D^\perp$ and each unit vector Z in $D^\perp$ perpendicular to W, we have on using Lemma 2.2.4

$$\begin{aligned} g(\eta, JW) &= g(h(W, Z), JW) \\ &= g(h(Z, W), JZ) \\ &= g(Z, W)\ g(\eta, JZ) \\ &= 0 \end{aligned} \tag{3.2.16}$$

Since M is assumed to be anti-holomorphic, (3.2.16) implies either $p = 1$ or

$$h(D^\perp, D^\perp) = 0. \tag{3.2.17}$$

Hence, (3.2.11), (3.2.13) and (3.2.17) imply that the equality case of (3.2.2) holds whenever $p > 1$.

When $p = 1$, M is a real hypersurface of $\bar{M}$. In this case, the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature if and only if (3.2.17) holds. So, in this case we also have equality case of (3.2.2) if the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature. From the first condition in (3.2.12), we also know that condition (3.2.17) holds if and only if M is minimal in $\bar{M}$.

By applying statement (ii), the converse is easy to verify.

For CR-warped products in complex space forms, we have the following.

Theorem 3.2.2. [18] *Let $M = N_T \times_f N_\perp$ be a non-trivial CR-warped product satisfying $\|\sigma\|^2 = 2p\|\nabla lnf\|^2$ in a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c. We have*

(a)N_T is a totally geodesic holomorphic submanifold of $\bar{M}$. Hence N_T is a complex space form $N^h(c)$ of constant holomorphic sectional curvature c.

(b)$N_\perp$ is a totally umbilical totally real submanifold of $\bar{M}(c)$. Hence, $N_\perp$ is a real space form of constant sectional curvature, say $\epsilon > c/4$.

(c)When $p = dimN_\perp > 1$, the warping function f satisfies $\|\nabla f\|^2 = \epsilon - (c/4)f^2$.

Proof. Under the hypothesis, we have (3.2.12). Statement (a) follows from the first equation of (3.2.12) and the fact that N_T is totally geodesic in M.

From the second equation of (3.2.12) and that $N_\perp$ is totally umbilical in M, we know that $N_\perp$ is totally umbilical in $\bar{M}(4c)$. Hence, by (1.2.13) and the equation of Gauss, we know that $N_\perp$ is of constant curvature, say $\epsilon \geq c/4$. From (3.2.10) we see that $\epsilon = c/4$ occurs only when the warping function is constant. Thus, we have statement (b).

Let $R^{N_\perp}$ denote the Riemann curvature tensor of $N_\perp$. Then we have

$$R(Z,W)V = R^{N_\perp}(Z,W)V - \|\nabla lnf\|^2(g(W,V)Z - g(Z,V)W), \tag{3.2.18}$$

for vectors Z, W, V tangent to $N_\perp$. By applying (1.2.13),(3.2.12),(3.2.18), the equation of Gauss, and statement (b), we obtain statement (c).

3.3. Warped Product Semi-slant Submanifolds in Kaehler Manifolds

In this section, we investigate semi-slant submanifolds in a Kaehler manifold $\bar{M}$ which are warped products of the form $N_\theta \times_f N_T$ (respectively $N_T \times_f N_\theta$), where N_θ is a proper slant submanifold and N_T is a holomorphic submanifold of $\bar{M}$.

Theorem 3.3.1. [49] *Let $\bar{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M = N_\theta \times_f N_T$ in $\bar{M}$ such that N_θ is a proper slant submanifold and N_T is a holomorphic submanifold of $\bar{M}$.*

Proof. By definition of semi-slant submanifolds, using statement (ii) of Proposition 1.2.1, we have

$$g(\nabla_{JX}Z, X) = Z(lnf)g(JX, X) = 0$$

for $X \in TN_T$ and $Z \in TN_\theta$. Thus taking into account that ∇ is a Levi-Civita connection, we obtain $g(\nabla_{JX}X, Z) = 0$. Using Gauss formula and Kaehler condition, we get $g(JZ, \bar{\nabla}_{JX}JX) = 0$. Then we derive $g(PZ + FZ, \bar{\nabla}_{JX}JX) = 0$. Thus, using Gauss formula, we have

$$-g(\nabla_{JX}PZ, JX) + g(FZ, h(JX, JX)) = 0.$$

Then from statement (ii) of Proposition 1.2.1, we obtain

$$g(FZ, h(JX, JX)) = PZ(lnf)g(X, X).$$

Thus, by polarization identity, we get

$$g(FZ, h(JX, JY)) = PZ(lnf)g(X, Y) \tag{3.3.1}$$

for $X, Y \in TN_T$ and $Z \in TN_\theta$. On the other hand, from Weingarten formula, we have $g(A_{FZ}JX, JY) = -g(\bar{\nabla}_{JX}FZ, JY)$. Since $\bar{\nabla}$ is a Levi-civita connection, we obtain $g(A_{FZ}JX, JY) = g(FZ, \bar{\nabla}_{JX}JY)$. Thus we get

$$g(A_{FZ}JX, JY) = g(Z, \bar{\nabla}_{JX}Y) - g(PZ, \bar{\nabla}_{JX}JY).$$

Thus we have

$$\begin{aligned} g(A_{FZ}JX, JY) &= g(Z, \nabla_{JX}Y) + g(\bar{\nabla}_{JX}PZ, JY) \\ &= -g(\nabla_{JX}Z, Y) + g(\nabla_{JX}PZ, JY) \\ &= -Z(lnf)g(JX, Y) + PZ(lnf)g(X, Y), \end{aligned}$$

therefore, we obtain

$$g(h(JX, JY), FZ) = -Z(lnf)g(JX, Y) + PZ(lnf)g(X, Y). \tag{3.3.2}$$

Thus (3.3.1) and (3.3.2) imply

$$Z(lnf)g(JX, Y) = 0$$

for $X, Y \in TN_T$ and $Z \in TN_\theta$. Since $N_T \neq \{0\}$ is a Riemannian and invariant submanifold, we obtain

$$Z(lnf) = 0$$

which shows that f is constant.

Theorem 3.3.1 shows that there do not exist warped product semi-slant submanifolds in the form $N_\theta \times_f N_T$ in Kaehler manifolds. Now, we are going to investigate warped product semi-slant submanifolds in the form $N_T \times_f N_\theta$ such that N_T is a holomorphic submanifold and N_θ is a proper slant submanifold of $\bar{M}$.

Theorem 3.3.2. [49] *Let $\bar{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M = N_T \times_f N_\theta$ in $\bar{M}$ such that N_T is a holomorphic submanifold and N_θ is a proper slant submanifold of $\bar{M}$.*

Proof. For $X \in TN_T$ and $Z \in TN_\theta$, from statement (ii) of Proposition 1.2.1, we have $g(\nabla_{PZ}X, Z) = X(lnf)g(PZ, Z) = 0$ due to $g(PZ, Z) = 0$. Since TN_T and TN_θ are orthogonal, we obtain $0 = g(\nabla_{PZ}X, Z) = -g(\nabla_{PZ}Z, X)$. Now we get $g(JX, \bar{\nabla}_{PZ}JZ) = 0$. Then

$$\begin{aligned} 0 &= g(JX, \bar{\nabla}_{PZ}PZ + FZ) \\ &= -g(\nabla_{PZ}JX, PZ) - g(JX, A_{FZ}PZ) \\ &= g(\nabla_{PZ}JX, PZ) + g(h(PZ, JX), FZ). \end{aligned}$$

Thus using Proposition 1.2.1 and 1.4.8, we get

$$JX(lnf)cos^2\theta g(Z, Z) = -g(h(PZ, JX), FZ) \tag{3.3.3}$$

for $X \in TN_T$ and $Z \in TN_\theta$. Substituting X by JX in (3.3.3) we arrive at

$$X(lnf)cos^2\theta g(Z, Z) = -g(h(PZ, X), FZ). \tag{3.3.4}$$

Also substituting Z by PZ in (3.3.4) and using (1.4.7) and (1.4.8), we obtain

$$g(h(P^2Z, X), FPZ) = -X(lnf)cos^2\theta g(PZ, PZ),$$

hence we have

$$g(h(Z, X), FPZ) = X(lnf)cos^2\theta g(Z, Z) \tag{3.3.5}$$

On the other hand, from Gauss formula $g(h(PZ, X), FW) = g(\bar{\nabla}_X PZ, FW)$ for $X \in TN_T$ and $Z, W \in TN_\theta$. Hence, $g(h(PZ, X), FW) = -g(PZ, \bar{\nabla}_X FW)$. Thus, we derive

$$\begin{aligned} g(h(PZ, X), FW) &= -g(PZ, \bar{\nabla}_X JW) + g(PZ, \bar{\nabla}_X PW) \\ &= g(JPZ, \bar{\nabla}_X W) + g(PZ, \nabla_X PW) \\ &= g(P^2Z, \bar{\nabla}_X W) + g(FPZ, h(X, W)) + g(PZ, \nabla_X PW), \end{aligned}$$

then, we obtain

$$g(h(PZ,X),FW) = -cos^2\theta X(lnf)g(Z,W)+g(FPZ,h(X,W))+X(lnf)g(PZ,PV$$

Also from (1.4.8), we get

$$g(h(PZ,X),FW) = -cos^2\theta X(lnf)g(Z,W)+g(FPZ,h(X,W))+X(lnf)g(Z,W)$$

Thus for $Z = W$ we have

$$g(h(PZ,X),FZ) = g(FPZ,h(X,Z)) \tag{3.3.6}$$

for $X \in TN_T$ and $Z \in TN_\theta$. Thus from (3.3.4),(3.3.5) and (3.3.6) we get

$$X(lnf)cos^2\theta g(Z,Z) = 0.$$

Since N_θ is a proper slant and Z is non-null, we obtain

$$X(lnf) = 0.$$

This implies that f is constant i.e., warped product is trivial.

CHAPTER 4

WARPED PRODUCT SUBMANIFOLDS IN NEARLY KAEHLER MANIFOLDS

In view of the applications of warped product submanifolds, recently the study of warped product submanifolds is extended to the settings of nearly Kaehler manifolds (cf. [36], [38], [39]). The investigations are relevant for the fact that a typical nearly Kaehler manifold S^6 does not admit CR-product submanifolds. This paved way to more general product submanifolds of S^6. The most natural candidate was a CR-warped product submanifold and that's how the first example of CR-warped product submanifold came up when K. Sekigawa [52] constructed a non-trivial CR-warped product submanifold in S^6. On the other hand, N. Ejiri [23] provided a categorical answer to a more general problem when he proved that "There exist countably many *immersions of* $S^1 \times S^{n-1}$ *into* S^{n+1} *such that the induced metric on* $S^1 \times S^{n-1}$ is a warped product metric of constant scaler curvature $n(n-1)$." The above theorem underlines the significance of the study of CR-warped product submanifolds in nearly Kaehler manifolds in general. The present chapter deals with the study of warped product submanifolds of nearly Kaehler manifolds in general.

4.1. CR-Submanifolds in Nearly Kaehler Manifolds

For CR-submanifolds of nearly Kaehler manifolds, we have the following theorems.

Theorem 4.1.1. [35] *The holomorphic distribution D on a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ is integrable if and only if*

$$\mathcal{Q}_X Y = 0 \quad \text{and} \quad h(X, JY) = h(JX, Y)$$

for each X, Y in D.

Proof. Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$. In view of the relation (2.1.5) (b) and Proposition 2.1.1, the necessary and sufficient condition for the holomorphic distribution D to be integrable on a CR-submanifold of a nearly Kaehler manifold, reduces to

$$2\mathcal{Q}_X Y = h(X, JY) - h(JX, Y) \tag{4.1.1}$$

for each $X, Y \in D$. Further, on using (p_4) and Lemma 1.3.1, we have

$$N^{\perp}(X,Y) = 8\mathcal{Q}_X JY \tag{4.1.2}$$

where $N^{\perp}$ denotes the normal part of the Nijenhuis tensor. On the other hand, by formula (1.3.8),

$$N^{\perp}(X,Y) = -2F([PX,Y] + [X,PY]) \tag{4.1.3}$$

from (4.1.2) and (4.1.3),

$$4\mathcal{Q}_X JY = -F([PX,Y] + [X,PY]). \tag{4.1.4}$$

It follows from (4.1.4) that D is integrable if and only if $\mathcal{Q}_X Y = 0$ for each $X, Y \in D$. Hence, by (4.1.1), the theorem follows.

Theorem 4.1.2. [35] *The totally real distribution $D^{\perp}$ on a CR-submanifold of a nearly Kaehler manifold is integrable if and only if*

$$g(\mathcal{P}_Z W, X) = 0,$$

or,

$$g(A_{JZ}W, X) = g(A_{JW}Z, X)$$

for each $Z, W \in D^{\perp}$ and $X \in D$.

Proof. By formula (2.1.1) and (2.1.5)(a), we get

$$2\mathcal{P}_Z W = P[W,Z] + A_{FZ}W - A_{FW}Z$$

for each $Z, W \in D^{\perp}$. Hence, $D^{\perp}$ is integrable if and only if

$$\mathcal{P}_Z W - A_{FZ}W + A_{FW}Z = 0. \tag{4.1.5}$$

Since for a nearly Kaehler manifold,

$$d\Omega(U,V,W) = -g((\bar{\nabla}_U J)V, W),$$

where Ω is the fundamental 2-form, (4.1.5) gives

$$2d\Omega(Z,W,X) = g(A_{JW}Z, X) - g(A_{JZ}W, X).$$

Further, as $\Omega(D, D^{\perp}) = 0 = \Omega(D^{\perp}, D^{\perp})$, we get

$$2g([Z,W], JX) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Thus, assertion is proved on taking account of (4.1.5).

4.2. CR-Submanifolds as Warped Product submanifolds in Nearly Kaehler Manifolds

First we consider warped product CR-submanifolds of the type $N_\perp \times_f N_T$ where N_T and $N_\perp$ are holomorphic and totally real submanifolds of a nearly Kaehler manifold $\bar{M}$. We have the following theorem which is an extension of the Theorem 3.1.1 to the setting of nearly Kaehler manifolds.

Theorem 4.2.1. [36] *There does not exist a proper warped product CR-submanifold $N_\perp \times_f N_T$ in nearly Kaehler manifolds.*

Proof. Let $\bar{M}$ be a nearly Kaehler manifold and $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of $\bar{M}$. By property (p_4) of $\mathcal{P}$, we have

$$\mathcal{P}_X JX + \mathcal{Q}_X JX = 0 \tag{4.2.1}$$

for each $X \in TN_T$.

By statement (ii) of Proposition 1.2.1, we have

$$\nabla_X Z = \nabla_Z X = (Zlnf)X. \tag{4.2.2}$$

for each $X, Y \in TN_T$ and $Z \in TN^\perp$. Hence,

$$g(\nabla_X Z, X) = (Zlnf)\|X\|^2 = g(\nabla_{JX} Z, JX). \tag{4.2.3}$$

On taking account of (1.2.3),(2.1.5) and (4.2.1), above equation can be written as

$$(Zlnf)\|X\|^2 = g(JZ, h(X, JX)). \tag{4.2.4}$$

Replacing X by JX in (4.2.4), we get

$$(Zlnf)\|X\|^2 = -g(JZ, h(X, JX)). \tag{4.2.5}$$

Thus from (4.2.4) and (4.2.5),

$$(Zlnf)\|X\|^2 = 0. \tag{4.2.6}$$

If M is assumed to be a proper CR-submanifold, then $Zlnf = 0$, i.e., M is simply a CR-product. This proves the Theorem.

In this section we shall study the warped product CR-submanifolds of the type $N_T \times_f N_\perp$ in a nearly Kaehler manifold $\bar{M}$. First, we shall discuss,

Lemma 4.2.1. [36] *Let M be a CR-warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then we have*

(i) $g(h(X,Y), JZ) = 0$

(ii) $g(\nabla_Z X, W) = X(\ln f) g(Z, W) = g(h(JX, Z), JW)$

for each X, $Y \in T(N_T)$ and Z, $W \in T(N_\perp)$.

Proof. By equations (2.1.1) and (2.1.2),

$$g(A_{FZ}X, Y) = g(\nabla_X Z, JY) - g(\mathcal{P}_X Z, Y).$$

The first term in the right hand side of the above equation is zero in view of Proposition 1.2.1. Thus, the equation reduces to

$$g(A_{FZ}X, Y) = -g(\mathcal{P}_X Z, Y).$$

The left hand side of the above equation is symmetric in X and Y whereas the right hand side is skew-symmetric in X and Y . That proves the statement (i), i.e.,

$$g(h(X,Y), JZ) = g(\mathcal{P}_X Z, Y) = 0$$

The first equality in (ii) is an immediate consequence of statement (ii) of Proposition 1.2.1. For the second equality, by Gauss formula, we may write

$$\begin{aligned} g(h(JX,Z), JW) &= g(\bar{\nabla}_Z JX, JW) \\ &= g(Q_Z X, JW) + g(\nabla_Z X, W) \\ &= g(Q_Z JX, W) + X(\ln f) g(Z, W). \\ &= -g(\mathcal{P}_Z W, JX) + X(\ln f) g(Z, W). \end{aligned}$$

The first term in the right hand side of the above equation is zero by virtue of Theorem 4.1.2 and the equation reduces to

$$g(h(JX,Z), JW) = (X \ln f) g(Z, W)$$

which completes the proof of statement (ii).

Theorem 4.2.2. [36] *Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ with integrable distributions D and $D^{\perp}$. Then M is locally a CR-warped product if and only if*

$$A_{JZ}X = -(JX\lambda)Z \tag{4.2.7}$$

for each $X \in D$, $Z \in D^{\perp}$ and λ, a C^{∞}-function on M such that $W\lambda = 0$ for each $W \in D^{\perp}$.

Proof. If M is a CR-warped product submanifold $N_T \times_f N_{\perp}$, then on applying Lemma 4.2.1, we obtain (4.2.7). In this case $\lambda = \ln f$.

Conversely, suppose $A_{JZ}X = -(JX\lambda)Z$, then

$$g(h(X,Y), JZ) = 0$$

i.e., $h(X,Y) \in \mu$, for each $X,\ Y \in D$. As D is assumed to be integrable, by Theorem 4.1.1, $Q_XY = 0$ and therefore by (2.1.2)

$$F\nabla_XY = h(X, JY) - fh(X,Y).$$

As $h(X,Y) \in \mu$ for each $X,\ Y \in D$, $FU \in JD^{\perp}$ for each $U \in TM$ and $f\xi \in \mu$ for all $\xi \in T^{\perp}M$, we deduce from the above equation that $\nabla_XY \in D$. That means, leaves of D are totally geodesic in M. Now,

$$\begin{aligned} g(\nabla_ZW, X) &= g(J\bar{\nabla}_ZW, JX) \\ &= -g(\mathcal{P}_ZW, JX) - g(A_{JW}Z, JX). \end{aligned}$$

The first term in the right hand side of the above equation vanishes in view of Theorem 4.1.2 and the second term on making use of (4.2.7) reduces to $-X\lambda\, g(Z,W)$. That is, we have

$$g(\nabla_ZW, X) = -X\lambda\, g(Z,W). \tag{4.2.8}$$

Now, by Gauss formula

$$g(h^{\perp}(Z,W), X) = g(\nabla_ZW, X)$$

where $h^{\perp}$ denotes the second fundamental form of the immersion of $N_{\perp}$ into M. Using (4.2.8), the last equation gives

$$g(h^{\perp}(Z,W), X) = -X\lambda\, g(Z,W),$$

which shows that each leaf $N_{\perp}$ of $D^{\perp}$ is totally umbilical in M. Moreover, the fact that $W\lambda = 0$, for all $W \in D^{\perp}$, implies that the mean curvature vector on $N_\perp$ is parallel along $N_\perp$ i.e., each leaf of $D^\perp$ is an extrinsic sphere in M. Thus M is locally a warped product $N_T \times_f N_\perp$ of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$ of M (cf. Theorem 1.2.2). Here N_T is a leaf of D and $N_\perp$ is a leaf of $D^\perp$ and f is a warping function.

Now, we give an example showing that proper CR-warped product submanifolds $N_T \times {}_f N_\perp$ do exist in nearly Kaehler manifold. This example was given by K. Sekigawa [52].

Example 4.2.1. Let $S^2 = \{y = (y_2, y_4, y_6) \in \mathbb{R}^3;\ y_2^2 + y_4^2 + y_6^2 = 1\}$ be a unit 2-sphere and $S^1 = \{z = e^{it},\ t \in \mathbb{R}\}$ a unit circle. Let ψ be the C^∞-mapping from the product manifold $S^2 \times S^1$ into S^6 i.e.,

$$\psi : S^2 \times S^1 \longrightarrow S^6$$

defined as

$$\begin{aligned}\psi(y, z) &= \psi((y_2, y_4, y_6), e^{it}) \\ &= (y_2 \cos t)e_2 - (y_2 \sin t)e_3 + (y_4 \cos 2t)e_4 + (y_4 \sin 2t)e_5 \\ &\quad + (y_6 \cos t)e_6 - (y_6 \sin t)e_7\end{aligned}$$

for $y = (y_2, y_4, y_6) \in S^2$ and $z = e^{it} \in S^1$, $t \in \mathbb{R}$. Then we may easily check that ψ gives rise to an isometric immersion from the warped product Riemannian manifold $S^2 \times {}_f S^1$ into S^6, where f is the warping function on S^2 which is given by the restriction of the function F on $\mathbb{R}^3$ defined as $F(y_2, y_4, y_6) = \sqrt{(1 + 3y_4^2)}$. This is an example of 3-dimensional proper CR-warped product submanifolds of S^6 such that both the holomorphic distribution and totally real distribution are integrable.

4.3. Semi-slant Warped Product Submanifolds of Nearly Kaehler Manifolds

Here we consider warped product submanifolds $M = N_T \times_f N_\theta$ of a nearly Kaehler manifold $\bar{M}$ where N_T and N_θ are holomorphic and slant submanifolds of $\bar{M}$ of real dimensions $2p$ and q respectively. These submanifolds are called *semi-slant warped product submanifolds.* A non-trivial warped product semi-slant submanifold $N_\theta \times_f N_T$ in $\bar{M}$ is non-existent in view of the following theorem:

Theorem 4.3.1. [37] *Let $\bar{M}$ be a nearly Kaehler manifold and $M = N \times_f N_T$ a warped product submanifold of $\bar{M}$ with N and N_T a Riemannian* *and a holomorphic submanifold of $\bar{M}$. Then M is trivial i.e., M is locally a Riemannian product of N and N_T.*

The following Lemma exhibits some formulae on a semi-slant warped product submanifold of a nearly Kaehler manifold.

Lemma 4.3.1. [39] *On a proper semi-slant warped product submanifold $M = N_T \times_f N_\theta$ of a nearly Kaehler manifold $\bar{M}$. We have*

(i) $g(h(X,Y), FZ) = 0$

(ii) $g(h(PX,Z), FZ) = (Xlnf)\|Z\|^2$

(iii) $g(h(X,Z), FW) - g(h(X,W), FZ) = \frac{2}{3}(Xlnf)g(PZ,W)$

(iv) $g(\mathcal{P}_X Y, Z) = g(\mathcal{Q}_X Y, FZ) = 0$

(v) $g(\mathcal{Q}_Z X, FZ) = \frac{2}{3}\cos^2\theta(Xlnf)\|Z\|^2$

for each $X, Y \in D$ and $Z \in D^\theta$.

Proof. As N_T is totally geodesic in M, $(\bar{\nabla}_X P)Y \in D$ and therefore by formula (2.1.1),

$$\begin{aligned} g(\mathcal{P}_X Y, Z) &= -g(th(X,Y), Z) \\ &= g(h(X,Y), FZ) \end{aligned}$$

The left hand side is skew symmetric whereas the right hand side is symmetric in X and Y in the above equation. That means

$$g(h(X,Y), FZ) = g(\mathcal{P}_X Y, Z) = 0.$$

This proves the statement (i).

Now, by (2.1.5)(a) $\mathcal{P}_U V + \mathcal{P}_V U = 0$. Using this fact in (2.1.1), we get

$$0 = (\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X - 2th(X,Z) - A_{FZ}X.$$

Further, since $(\bar{\nabla}_X P)Z = 0$ due to (1.4.3) and Proposition 1.2.1, whereas applying the same formulae on $(\bar{\nabla}_Z P)X$, the above equation takes the form

$$(PXlnf)Z - (Xlnf)PZ = 2th(X,Z) + A_{FZ}X. \tag{4.3.1}$$

Taking product with Z in both sides of (4.3.1) yields (ii).

It follows from the statement (ii) that

$$g(h(PX, Z), FW) + g(h(PX, W), FZ) = 2(Xlnf)g(Z, W).$$

for any $X \in D$ and $Z, W \in D^\theta$. Therefore, for orthogonal vector fields $Z, W \in D^\theta$

$$g(h(X, Z), FW) + g(h(X, W), FZ) = 0. \tag{4.3.2}$$

Now, taking product with $W \in D^\theta$ in (4.3.1) gives

$$(PXlnf)g(Z, W) - (Xlnf)g(PZ, W) = -2g(h(X, Z), FW) + g(h(X, W), FZ).$$

Interchanging Z and W and substracting the obtained equation from the above, we get

$$g(h(X, Z), FW) - g(h(X, W), FZ) = \frac{2}{3}(Xlnf)g(PZ, W).$$

This proves the statement (iii) of the Lemma.

For orthogonal vector fields Z and W, it can be deduced from (4.3.2) and the last equation that

$$g(h(X, W), FZ) = -g(h(X, Z), FW) = -\frac{1}{3}(Xlnf)g(PZ, W) \tag{4.3.3}$$

In particular for $W = PZ$, the above formula on taking account of (1.4.8) yields

$$g(h(X, PZ), FZ) = -g(h(X, Z), FPZ) = -\frac{1}{3}\cos^2\theta(Xlnf)\|Z\|^2 \tag{4.3.4}$$

By (2.1.2) and Proposition 1.2.1

$$g(\mathcal{Q}_X Y, FZ) = g(h(X, PY), FZ) - g(fh(X, Y), FZ).$$

The two terms in the right hand side of the above equation are zero by statement (i) of the Lemma and formula (2.2.15). That means

$$g(\mathcal{Q}_X Y, FZ) = 0.$$

Thus, we conclude that

$$g(\mathcal{P}_X Y, Z) = g(\mathcal{Q}_X Y, FZ) = 0.$$

This proves the statement (iv) of the Lemma.

Now, by (2.1.2) and Proposition 1.2.1, we may write

$$h(JX, Z) = \mathcal{Q}_Z X + (Xlnf)FZ + fh(X, Z). \tag{4.3.5}$$

Taking product with FZ in (4.3.5) and using the formulae (4.3.3), (1.4.8), (2.2.15) and statement (i) and (ii) of the Lemma, we obtain

$$g(\mathcal{Q}_Z X, FZ) = \frac{2}{3}\cos^2\theta(Xlnf)\|Z\|^2 \tag{4.3.6}$$

This completes the proof of the Lemma. □

Let $\{X_1, X_2, ..., X_p, X_{p+1} = JX_1, ..., X_{2p} = JX_p\}$ and $\{Z_1, Z_2, ..., Z_{\frac{q}{2}}, Z_{\frac{q}{2}+1} = \frac{PZ_1}{cos\theta}, ..., Z_q = \frac{PZ_{\frac{q}{2}}}{cos\theta}\}$ be local frames of orthonormal vector fields on N_T and N_θ respectively. Then, for $X \in D$,

$$\begin{aligned}\sum_{r,s=1}^{q} g(h(X, Z_s), FZ_r)^2 &= \sum_{r=1}^{q} g(h(X, Z_r), FZ_r)^2 + \sum_{r\neq s} g(h(X, Z_s), FZ_r)^2\\ &= 2q(PXlnf)^2 + \frac{1}{cos^2\theta}\sum_{r=1}^{q/2}[g(h(X, Z_r), FPZ_r)^2\\ &\quad +g(h(X, PZ_r), FZ_r)^2].\end{aligned}$$

It can be noted that $g(h(X, Z_s), FZ_r) = 0$ for $Z_s \neq Z_r, PZ_r$ due to formula (4.3.3). Now, using formula (4.3.4), the above equation takes the form

$$\sum_{r,s=1}^{q} g(h(X, Z_s), FZ_r)^2 = 2q(PXlnf)^2 + \frac{2q}{9}\cos^2\theta(Xlnf)^2.$$

Replacing X by a basic vector field X_i and taking summation over i in both sides of the above equation while making use of the formula (1.3.1), we get

$$\sum_{i=1}^{2p}\sum_{r,s=1}^{q} g(h(X_i, Z_s), FZ_r)^2 = 2q(1 + \frac{1}{9}\cos^2\theta)\|\nabla lnf\|^2. \tag{4.3.7}$$

Let us denote $\sum_{i=1}^{2p}\sum_{r=1}^{q} \|h_{FD^\theta}(X_i, Z_r)\|$ by $\|h_{FD^\theta}(D, D^\theta)\|$ and $\sum_{i=1}^{2p}\sum_{r=1}^{q} \|h_\mu(X_i, Z_r)\|$ by $\|h_\mu(D, D^\theta)\|$. Then we obtain

Theorem 4.3.2. [39] *Let M be a semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then*

$$\|h_{FD^\theta}((D, D^\theta)\|^2 = 2q\csc^2\theta(1+\frac{1}{9}\cos^2\theta)\|\nabla lnf\|^2 \tag{4.3.8}$$

and

$$\|fh_{FD^\theta}((D, D^\theta)\|^2 = 2q\cot^2\theta(1+\frac{1}{9}\cos^2\theta)\|\nabla lnf\|^2 \tag{4.3.9}$$

Proof. For any $X \in D$ and $Z \in D^\theta$, by (2.2.13)

$$\|h_{FD^\theta}(X,Z)\|^2 = g(h_{FD^\theta}(X,Z), h_{FD^\theta}(X,Z)) = \csc^2\theta\sum_r g(h(X,Z), FZ_r)^2.$$

Replacing X and Z by basic vector fields in the above expression and taking summation over $i = 1, 2, ...2p$ and $s = 1, 2, ...q$, we get

$$\|h_{FD^\theta}(D, D^\theta)\|^2 = \sum_{i=1}^{2p}\sum_{r,s=1}^{q}\csc^2\theta g(h(X_i, Z_s), FZ_r)^2$$

Using (4.3.7), the right hand side of the above equation is $2q\csc^2\theta(1+\frac{1}{9}\cos^2\theta)\|\nabla lnf\|^2$.
Hence,

$$\|h_{FD^\theta}(D, D^\theta)\|^2 = 2q\csc^2\theta(1+\frac{1}{9}\cos^2\theta)\|\nabla lnf\|^2.$$

This proves (4.3.8). Similarly, replacing U and V by basic vector fields X_i and Z_s respectively in (2.2.14) and taking summation over i, r and s, we get

$$\|fh_{FD^\theta}(D, D^\theta)\|^2 = \cot^2\theta\sum_{i=1}^{2p}\sum_{r,s=1}^{q} g(h(X_i, Z_s), FZ_r)^2.$$

Making use of Lemma 4.3.1, formulae (4.3.3) and (4.3.7) on the above equation, we obtain (4.3.9). □

Now, before finding an estimate for $\|h_\mu(D, D^\theta)\|$, we prove the following

Lemma 4.3.2. [39] *On a semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$*

$$\sum_{i=1}^{p}\Big[\sum_{r,s=1}^{q} g(h(PX_i,Z_s),FZ_r).g(h(X_i,PZ_s),FZ_r)$$
$$- g(h(X_i,Z_s),FZ_r).g(h(PX_i,PZ_s),FZ_r)\Big]$$

$$= -\frac{4q}{3}\cos^2\theta\|\nabla lnf\|^2 \tag{4.3.10}$$

where $\{X_1, X_2, ..., X_p, PX_1, ..., PX_p\}$ *and* $\{Z_1, Z_2, ..., Z_{q/2}, \frac{PZ_1}{cos\theta}, ..., \frac{PZ_{q/2}}{cos\theta}\}$ *are adapted frames of orthonormal vector fields on* N_T *and* N_θ *respectively.*

Proof. For $X \in D$ and $Z \in D^\theta$, we have

$$\sum_{i=1}^{p}\Big[\sum_{r,s=1}^{q} g(h(PX_i,Z_s),FZ_r).g(h(X_i,PZ_s),FZ_r)\Big]$$

$$=\sum_{i=1}^{p}\Big[\sum_{r=1}^{q} g(h(PX_i,Z_r),FZ_r)g(h(X_i,PZ_r),FZ_r)$$
$$+\sum_{r\neq s} g(h(PX_i,Z_s),FZ_r)g(h(X_i,PZ_s),FZ_r)\Big].$$

$$=\sum_{i=1}^{p}\Big[\sum_{r=1}^{q} g(h(PX_i,Z_r),FZ_r).g(h(X_i,PZ_r),FZ_r)$$
$$+\sum_{r=1}^{q/2} g(h(PX_i,Z_r),FZ_{r+\frac{q}{2}}).g(h(X_i,PZ_r),FZ_{r+\frac{q}{2}}).$$
$$+\sum_{r=1}^{q/2} g(h(PX_i,\tfrac{PZ_r}{cos\theta}),FZ_r).g(h(X_i,\tfrac{P^2Z_r}{cos\theta}),FZ_r)\Big].$$

The terms containing Z_s and Z_r in the series with $Z_s \neq Z_r, PZ_r$ vanish in view of formula (4.3.3). Now using (4.3.4), Lemma 4.3.1 (*ii*), (1.4.8) and (1.4.9), the above summation takes the form

$$\sum_{i=1}^{p}[-\frac{2q}{3}\cos^2\theta(X_ilnf)^2 - \frac{2q}{3}\cos^2\theta(PX_ilnf)^2] \tag{4.3.11}$$
$$= -\frac{2q}{3}\cos^2\theta\sum_{i=1}^{p}[(X_ilnf)^2 + (PX_ilnf)^2].$$

Hence, by using (1.3.1) in the above expression, we get

$$\sum_{i=1}^{p}[\sum_{r=1}^{q} g(h(PX_i, Z_s), FZ_r).g(h(X_i, PZ_s), FZ_r)] \tag{4.3.12}$$

$$= -\frac{2q}{3}\cos^2\theta\|\nabla lnf\|^2.$$

Since the second term in the left hand side of (4.3.10) can be obtained from the first one on replacing $X_i'^s$ by $-PX_i'^s$ and the terms in (4.3.11) are square terms, it follows that

$$g(h(PX_i, Z_s), FZ_r).g(h(X_i, PZ_s), FZ_r) \tag{4.3.13}$$

$$= \frac{2q}{3}\cos^2\theta\|\nabla lnf\|^2.$$

However, (4.3.13) can also be proved on the same lines as (4.3.12). Now, from (4.3.12) and (4.3.13), we obtain (4.3.10).

Now, on a semi-slant warped product submanifold of a generalized complex space form, we prove

Theorem 4.3.3. [39] *Let $M = N_T \times_f N_\theta$ be a semi-slant warped product submanifold of a generalized complex space form $\bar{M}(c, \alpha)$. Then, we have*

$$\|h\|^2 \geq q\Delta lnf + (\frac{q}{3}\cos^2\theta(5 - 6csc^2\theta) + 2q)\|\nabla lnf\|^2 \tag{4.3.14}$$

$$+\frac{pq(c-\alpha)}{2}\sin^2\theta + \|\mathcal{Q}_D D^\theta\|^2,$$

where h denotes the second fundamental form of the immersion of M into $\bar{M}(c,\alpha)$, Δlnf is the Laplacian of lnf, ∇lnf is the gradient of lnf, $2p$ and q are the real dimensions of N_T and N_θ respectively.

If the equality sign in (4.3.14) holds identically, then N_T is a totally geodesic submanifold and N_θ is a totally umbilical submanifold of $\bar{M}$. Moreover, in this case

$$g(h_\mu(PX, Z), h_\mu(X, PZ)) = g(h_\mu(X, Z), h_\mu(PX, PZ))$$

for each $X \in D$ and $Z \in D^\theta$.

Proof. For $X \in D$ and $Z \in D^\theta$, the equation of Coddazi is written as

$$\tilde{R}(X,JX,Z,FZ) = g(\nabla^{\perp}_{X}h(JX,Z),FZ) - g(\nabla^{\perp}_{JX}h(X,Z),FZ)$$
$$+g(h(\nabla_{JX}X,Z),FZ) - g(h(\nabla_{X}JX,Z),FZ)$$
$$+g(h(X,\nabla_{JX}Z),FZ) - g(h(JX,\nabla_{X}Z),FZ) \tag{4.3.15}$$

where,

$$g(\nabla^{\perp}_{X}h(JX,Z),FZ) = X.g(h(JX,Z),FZ) - g(h(JX,Z),\nabla^{\perp}_{X}FZ) \tag{4.3.16}$$

In view of formula (ii) of Lemma 4.3.1,

$$X.g(h(JX,Z),FZ) = \{X.(Xlnf) + 2(Xlnf)^2\}\|Z_r\|^2 \tag{4.3.17}$$

whereas on applying (4.3.5) and (2.1.2), the second term in the right hand side of (4.3.16) is expressed as

$$g(h(JX,Z),\nabla^{\perp}_{X}FZ) = -\|\mathcal{Q}_X Z\|^2 + (Xlnf)g(\mathcal{Q}_X Z,FZ)$$
$$+(Xlnf)^2\|Z\|^2 + (Xlnf)g(FZ,fh(X,Z))$$
$$-g(h(PX,Z),h(X,PZ)) + g(fh(X,Z),fh(X,Z))$$

which on using (4.3.6), (2.2.15) and Lemma 4.3.1 (ii), reduces to

$$g(h(JX,Z),\nabla^{\perp}_{X}FZ) = -\|\mathcal{Q}_X Z\|^2 + (1-\tfrac{1}{3}\cos^2\theta)(Xlnf)^2\|Z\|^2$$
$$-g(h(PX,Z),h(X,PZ))$$
$$+g(fh(X,Z),fh(X,Z)). \tag{4.3.18}$$

On substitution from (4.3.17) and (4.3.18), equation (4.3.16) takes the form

$$g(\nabla^{\perp}_{X}h(JX,Z),FZ) = X(Xlnf)\|Z\|^2 + (1+\tfrac{1}{3}\cos^2\theta)(Xlnf)^2\|Z\|^2$$
$$+\|\mathcal{Q}_X Z\|^2 + g(h(PX,Z),h(X,PZ))$$
$$-g(fh(X,Z),fh(X,Z)), \tag{4.3.19}$$

where

$$g(h(PX,Z),h(X,PZ)) = g(h_{FD^\theta}(PX,Z),h_{FD^\theta}(X,PZ)) + g(h_\mu(PX,Z),h_\mu(X,PZ))$$

which on taking account of (2.2.12), is simplified as

$$g(h(PX,Z),h(X,PZ)) = \csc^2 \sum_r g(h(PX,Z),FZ_r)g(h(X,PZ),FZ_r)$$
$$(4.3.20) \qquad +g(h_\mu(PX,Z),h_\mu(X,PZ))$$

where $\{Z_1, Z_2, ..., Z_q\}$ is an orthonormal frame of vector fields on N_θ. On the other hand,

$$g(fh(X,Z),fh(X,Z)) = \|fh_{FD^\theta}(X,Z)\|^2 + \|h_\mu(X,Z)\|^2$$

and by (2.2.14)

$$(4.3.21) \qquad \|fh_{FD^\theta}(X,Z)\|^2 = \cot^2\theta \sum_r g(h(X,Z),FZ_r)^2.$$

Taking account of (4.3.20) and (4.3.21), equation (4.3.19) takes the form

$$\begin{aligned} g(\nabla_X^\perp h(JX,Z),FZ) &= X(Xlnf)\|Z\|^2 + (1+\tfrac{1}{3}\cos^2\theta)(Xlnf)^2\|Z\|^2 \\ &+ \csc^2\theta \sum_r g(h(PX,Z),FZ_r)g(h(X,PZ),FZ_r) \\ &+ g(h_\mu(PX,Z),h_\mu(X,PZ)) - \|h_\mu(X,Z)\|^2 \\ (4.3.22) \qquad &- \cot^2\theta \sum_r g(h(X,Z),FZ_r)^2 + \|\mathcal{Q}_X Z\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} g(\nabla_{JX}^\perp h(X,Z),FZ) &= -JX(JXlnf)\|Z\|^2 - (1+\tfrac{1}{3}\cos^2\theta)(JXlnf)^2\|Z\|^2 \\ &+ \csc^2\theta \sum_r g(h(X,Z),FZ_r)g(h(PX,PZ),FZ_r) \\ &+ g(h_\mu(X,Z),h_\mu(PX,PZ)) + \|h_\mu(JX,Z)\|^2 \\ &+ \cot^2\theta \sum_r g(h(PX,Z),FZ_r)^2 \\ (4.3.23) \qquad &- \|\mathcal{Q}_{JX} Z\|^2. \end{aligned}$$

Now, taking account of Lemma 4.3.1 (*ii*) and the fact that N_T is totally geodesic in M, we get

$$\begin{aligned} g(h(\nabla_X JX, Z), FZ) &= -(J\nabla_X JX) lnf g(Z,Z) \\ &= -g(\nabla_Z J\nabla_X JX, Z) \\ &= -g(\nabla_X JX, J\nabla_Z Z) \\ &= -(g(\nabla_X X, \nabla_Z Z) + g(Jh(X,JX), \nabla_Z Z)). \end{aligned}$$

Similarly,

$$\begin{aligned} g(h(\nabla_{JX} X, Z), FZ) &= -(J\nabla_{JX} X) lnf g(Z,Z) \\ &= -(g(\nabla_{JX} JX, \nabla_Z Z) + g(Jh(X,JX), \nabla_Z Z)). \end{aligned}$$

Thus

$$\begin{aligned} & g(h(\nabla_{JX} X, Z), FZ) - g(h(\nabla_X JX, Z), FZ) \\ &= g(\nabla_X X, \nabla_Z Z) + g(\nabla_{JX} JX, \nabla_Z Z) \\ &= -\{(\nabla_X X lnf) + (\nabla_{JX} JX lnf)\} \|Z\|^2). \end{aligned} \tag{4.3.24}$$

On the other hand, by Proposition 1.2.1 and Lemma 4.3.1 (*ii*), we have

$$\begin{aligned} & g(h(X, \nabla_{JX} Z), FZ) - g(h(JX, \nabla_X Z), FZ) \\ &= -((X lnf)^2 + (JX lnf)^2) \|Z\|^2. \end{aligned} \tag{4.3.25}$$

Now, substituting the values from (4.3.22), (4.3.23), (4.3.24) and (4.3.25) into (4.3.15), we get

$$\begin{aligned}
\bar{R}(X,JX,Z,FZ) &= \{X(Xlnf)+JX(JXlnf)\\
&-(\nabla_X X)lnf+(\nabla_{JX}JX)lnf\}\|Z\|^2\\
&+\tfrac{1}{3}\cos^2\theta((Xlnf)^2+(JXlnf)^2)\|Z\|^2\\
&+\csc^2\theta\sum_r\{g(h(PX,Z),FZ_r)g(h(X,PZ),FZ_r)\\
&-g(h(X,Z),FZ_r)g(h(PX,PZ),FZ_r)\}\|Z\|^2\\
&-\cot^2\theta\sum_r\{g(h(X,Z),FZ_r)^2+g(h(PX,Z),FZ_r)^2\}\|Z_r\|^2\\
&+g(h_\mu(PX,Z),h_\mu(X,PZ))-g(h_\mu(X,Z),h_\mu(PX,PZ))\\
&-\|h_\mu(X,Z)\|^2-\|h_\mu(JX,Z)\|^2+\|\mathcal{Q}_X Z\|^2+\|\mathcal{Q}_{JX}Z\|^2.
\end{aligned}$$

Choosing X, Z as basic vector fields on N_T and N_θ, and evaluating $\bar{R}(X,JX,Z,FZ)$ through Coddazi equation, we get

$$\begin{aligned}
\bar{R}(X_i,JX_i,Z_r,FZ_r) &= [X_i(X_ilnf)+JX_i(JX_ilnf)\\
&-(\nabla_{X_i}X_i)lnf-(\nabla_{JX_i}JX_i)lnf]\|Z_r\|^2\\
&+\tfrac{1}{3}cos^2\theta[(X_ilnf)^2+(JX_ilnf)^2]\|Z_r\|^2\\
&+csc^2\theta\sum_r[(g(h(PX_i,Z_s),FZ_r)g(h(X_i,PZ_s),FZ_r)\\
&-(g(h(X_i,Z_s),FZ_r)g(h(PX_i,PZ_s),FZ_r)]\|Z_r\|^2\\
&-cot^2\theta\sum_r[(g(h(X_i,Z_s),FZ_r)^2+g(h(PX_i,Z_s),FZ_r)^2]\|Z_r\|^2\\
&+g(h_\mu(PX_i,Z_s),h_\mu(X_i,PZ_s))-g(h_\mu(X_i,Z_s),h_\mu(PX_i,PZ_s))\\
&-\|h_\mu(X_i,Z_r)\|^2-\|h_\mu(JX_i,Z_r)\|^2+\|\mathcal{Q}_{X_i}Z_r\|^2+\|\mathcal{Q}_{JX_i}Z_r\|^2.
\end{aligned}$$

On the other hand, by formula (1.3.17)

$$\sum_{i=1}^{2p}\sum_{r=1}^{q}\bar{R}(X_i,JX_i,Z_r,FZ_r)=\frac{pq(\alpha-c)}{2}sin^2\theta.$$

Making use of this fact, together with (1.3.1), (1.3.2), (1.3.17), (4.3.7) and (4.3.10), while taking summation over $i = 1, 2, \ldots, p$; $r, s = 1, 2, \ldots, q$, we obtain

$$\begin{aligned} \|h_\mu(D, D^\theta)\|^2 &\geq q\Delta lnf + \tfrac{q}{3}\cos^2\theta\|\nabla lnf\|^2 - \tfrac{4q}{3}\cot^2\theta\|\nabla lnf\|^2 \\ &\quad -2qcot^2\theta(1 + \tfrac{1}{9}\cos^2\theta)\|\nabla lnf\|^2 + \|\mathcal{Q}_D D^\theta\|^2 \\ &\quad -\tfrac{pq(\alpha-c)}{2}\sin^2\theta, \end{aligned} \tag{4.3.26}$$

where we have denoted $\sum_{i=1}^{2p}\sum_{r=1}^{q}\|\mathcal{Q}_{X_i}Z_r\|^2$ by $\|\mathcal{Q}_D D^\theta\|^2$.

Now, combining (4.3.8) and (4.3.26), we arrive at

$$\begin{aligned} \|h\|^2 &\geq q\Delta lnf + \tfrac{q}{3}\cos^2\theta - \tfrac{4}{3}q\cot^2\theta + 2q(1 + \tfrac{\cos^2\theta}{9})\|\nabla lnf\|^2 \\ &\quad +\|\mathcal{Q}_D D^\theta\|^2 + \tfrac{pq(c-\alpha)}{2}\sin^2\theta. \end{aligned}$$

That is,

$$\begin{aligned} \|h\|^2 &\geq q\Delta lnf + (\tfrac{q}{3}\cos^2\theta(5 - 6csc^2\theta) + 2q)\|\nabla lnf\|^2 \\ &\quad +\tfrac{pq(c-\alpha)}{2}\sin^2\theta + \|\mathcal{Q}_D D^\theta\|^2, \end{aligned}$$

If the equality case of (4.3.14) holds identically, then

$$h(D, D) = 0, \qquad h(D^\theta, D^\theta) = 0, \tag{4.3.27}$$

and

$$g(h_\mu(PX, Z), h_\mu(X, PZ)) = g(h_\mu(X, Z), h_\mu(PX, PZ))$$

for each $X \in D$ and $Z \in D^\theta$.

Since N_T is totally geodesic in M the first term in (4.3.27) implies that N_T is totally geodesic in $\bar{M}$. Further, if h_2 is the second fundamental form of the immersion of N_θ in M, then by (1.2.3) and Proposition 1.2.1

$$h_2(Z, W) = -\nabla lnf g(Z, W) \tag{4.3.28}$$

for each $Z, W \in D^\theta$. Equation (4.3.28) and the second relation in (4.3.27) imply that N_θ is totally umbilical in $\bar{M}$. $\square$

Corollary 4.3.1. *On a CR-warped product submanifold of a generalized complex space form $\bar{M}(c,\alpha)$, the squared norm of second fundamental form satisfies*

$$\|h\|^2 \geq q\Delta lnf + 2q\|\nabla lnf\|^2 + \|\mathcal{Q}_D D^\perp\|^2 + \tfrac{pq(c-\alpha)}{2}.$$

Corollary 4.3.2. *On a CR-warped product submanifold of S^6*

$$\|h\|^2 \geq q\{\Delta lnf + 2\|\nabla lnf\|^2\} + \|\mathcal{Q}_D D^\perp\|^2.$$

Corollary 4.3.3. *On a CR-warped product submanifold of a complex space form*

$$\|h\|^2 \geq q\{\Delta lnf + 2\|\nabla lnf\|^2\}.$$

The inequalities in Corollaries 4.3.1 and 4.3.3 are obtained in [21].

CHAPTER 5

WARPED PRODUCT SUBMANIFOLDS OF AN l.C.K MANIFOLD

K. Matsumoto [42], V. Bonanzinga [9] and M.I. Munteanu [46] extended the study of CR-warped product submanifolds to the setting of l.c.K. manifolds. Various important formulae and geometric properties of the submanifold are obtained by them. As a step forward, V. Khan and N. Jamal [32], studied generic warped product submanifolds of l.c.K. manifolds and worked out a method of finding a non-trivial generic warped product submanifold. An account of these studies is given in the present chapter.

5.1. Warped Product CR-submanifolds in l.c.K. manifolds

Proposition 5.1.1. [33] *A Hermitian manifold $\bar{M}$ with structure (J, g) is l.c.K. if and only if there exists a global 1-form α, called the Lee form, satisfying*

$$d\alpha = 0 \tag{5.1.1}$$

$$(\bar{\nabla}_Y J)X = -g(\alpha^\sharp, X)JY + g(X,Y)\beta^\sharp + g(JX,Y)\alpha^\sharp - g(\beta^\sharp, X)Y \tag{5.1.2}$$

for any $X, Y \in T\bar{M}$, where $\bar{\nabla}$ is the covariant differentiation with respect to g, $\alpha^\sharp$ is the dual vector field of α , the 1-form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β.

On a CR-submanifold, the integrability conditions for the canonical distributions were obtained in [43] as

Proposition 5.1.2. *On a CR-submanifold of a locally conformal Kaehler manifold,*
(i) $D^\perp$ is involutive.
(ii) D is involutive if and only if

$$g(h(X, JY) - h(JX, Y) + g(X, JY)\alpha^\sharp, FZ) = 0,$$

for all X, Y in D and Z in $D^{\perp}$.

Let N_T (resp. $N_{\perp}$) be a holomorphic (resp. totally real) submanifold in an l.c.K. manifold $\bar{M}$. We consider a warped product submanifold of the form $M = N_{\perp} \times_f N_T$ with a warping function $f(>0) \in C^{\infty}(N_{\perp})$. We call such a manifold a *warped product CR-submanifold* in an l.c.K. manifold $\bar{M}$.

By Corollary 1.2.1, $N_{\perp}$ is totally geodesic in M, that is, $\nabla_W Z \in TN_{\perp}$ for any $Z, W \in TN_{\perp}$. This means that

$$g(\nabla_W Z, X) = 0 \tag{5.1.3}$$

for any $X \in D$ and $Z, W \in D^{\perp}$, where we put $D = TN_T$ and $D^{\perp} = TN_{\perp}$.
Using (5.1.2) and (5.1.3), the formulae of Gauss and Weingarten yield

$$g(h(JX, Z), JW) = g(Z, W)g(\alpha^{\sharp}, X) \tag{5.1.4}$$

for any X in D and Z, W in $D^{\perp}$.
Now let h^T (resp. A^T) be the second fundamental form (resp. the shape operator) of N_T in M, then using Gauss formula and Proposition 1.2.1, we have

$$g(h^T(X, Y), Z) = g(A_Z^T X, Y) = -(Zlnf)g(X, Y) \tag{5.1.5}$$

for any X, Y in D and Z in $D^{\perp}$. The equation (5.1.5) means

$$h^T(X, Y) = -(\nabla lnf)g(X, Y) \tag{5.1.6}$$

for any X, Y in D where ∇lnf is the gradient of lnf. Thus we have

Proposition 5.1.3. [9] *On a warped product CR-submanifold $M = N_{\perp} \times_f N_T$ of an l.c.K. manifold $\bar{M}$, the holomorphic submanifold N_T is totally umbilical in M.*

Theorem 5.1.1. [9] *In a proper warped product CR-submanifold M of an l.c.K. manifold $\bar{M}$, if the Lee vector field $\alpha^{\sharp}$ is normal to $D^{\perp}$, then M is CR-product and trivial.*

Proof. Let $\hat{h}$ be the second fundamental form of N_T in $\bar{M}$ i.e.,

$$\bar{\nabla}_X Y = \nabla_X^T Y + \hat{h}(X, Y)$$

for any X, Y in D. Then we have

$$\hat{h}(X, Y) = h^T(X, Y) + h(X, Y) \tag{5.1.7}$$

for any X, Y in D. Using (5.1.5) and (5.1.7), we get

$$g(\hat{h}(X,Y),Z) = g(h^T(X,Y),Z) = -(Zlnf)g(X,Y) \tag{5.1.8}$$

for any $X, Y \in D$ and $Z \in D^\perp$. Since N_T is holomorphic, using (5.1.2) and (5.1.7), we get

$$g(\hat{h}(X,JY),U) = g(J\hat{h}(X,Y),U) + g(JX,Y)g(\alpha^\sharp,U) - g(X,Y)g(\beta^\sharp,U)$$

for any $X, Y \in D$ and $U \in D^\perp \oplus T^\perp M$. From the above equation, we get

$$g(\hat{h}(X,JY),U) = g(\hat{h}(Y,JX),U) + 2g(JX,Y)g(\alpha^\sharp,U) \tag{5.1.9}$$

In (5.1.9), we put $Y = JX$, then we have

$$-g(\hat{h}(X,X),U) = g(\hat{h}(JX,JX),U) + 2\|X\|^2 g(\alpha^\sharp,U) \tag{5.1.10}$$

for any $X \in D$ and $U \in D^\perp \oplus T^\perp M$.

In (5.1.10), if the vector field U is an element of $D^\perp$ (put it Z), then we have

$$-g(\hat{h}(X,X),Z) = g(\hat{h}(JX,JX),Z) + 2\|X\|^2 g(\alpha^\sharp,U)$$

for any $X \in D$ and $Z \in D^\perp$. From (5.1.9) and the above equation, we have

$$Zlnf = g(\alpha^\sharp,Z) \tag{5.1.11}$$

for any $Z \in D^\perp$ if M is proper.

5.2. CR-Warped Products in l.c.K. Manifolds

In this section, we consider CR-warped product submanifold $M = N_T \times_f N_\perp$ in an l.c.K. manifold $\bar{M}$. In a CR-warped product of an l.c.K. manifold $\bar{M}$, we prove

Proposition 5.2.1. [9] *In a proper CR-warped product $M = N_T \times_f N_\perp$ in an l.c.K. manifold $\bar{M}$, the Lee vector field $\alpha^\sharp$ is orthogonal to $D^\perp$.*

Proof. By virtue of (5.1.2) we have

$$g(h(X,Y),JZ) = g(\bar{\nabla}_X Y,JZ) = g(X,Y)g(\beta^\sharp,Z) + g(\alpha^\sharp,Z)g(JY,X)$$

for any $X, Y \in D$ and $Z \in D^\perp$. Since the left hand side of above equation is symmetric with respect to X and Y , we get $g(JX,Y)g(\alpha^\sharp,Z) = 0$ This means $g(\alpha^\sharp,Z) = 0$ for any $Z \in D^\perp$.

By virtue of the above proposition, we have

$$(5.2.1) \qquad g(h(X,Y),JZ) = g(A_{JZ}X,Y) = g(X,Y)g(\beta^\sharp, Z).$$

Especially, if the Lee vector field $\alpha^\sharp$ is tangent to M, then (5.2.1) means

$$(5.2.2) \qquad g(h(D,D),JD^\perp) = 0.$$

For any $X \in D$ and $Z, W \in D^\perp$, we have from (5.1.2) and Proposition 1.2.1

$$(5.2.3) \quad g(h(JX,Z),JW) = g(\bar{\nabla}_Z JX, JW) = (Xlnf - g(\alpha^\sharp, X))g(Z,W)$$

Next, we consider the case of $h(D,D^\perp) \subset JD^\perp$. Then we can prove that

$$(5.2.4) \qquad \nabla_X^\perp(JZ) = J\nabla_X Z$$

for any $X \in D$ and $Z \in D^\perp$.

The proof of (5.2.4) is as follows,

(*i*) $g(\nabla_X^\perp JZ, JW) = g(J\nabla_X Z, JW) = g(\nabla_X Z, W)$,

(*ii*) $g(\nabla_X^\perp JZ - J\nabla_X Z, \xi) = 0$ for any $X \in D, Z, W \in D^\perp$ and $\xi \in \mu$.

For (i), the left hand side of (i) is equal to

$g((\bar{\nabla}_X J)Z, JW) + g(J\bar{\nabla}_X Z, JW) = g(\bar{\nabla}_X Z, W)$=the right hand side.

Theorem 5.2.1. [32] *A CR-submanifold of an l.c.K. manifold $\bar{M}$ is a CR-warped product submanifold if and only if the Lee vector field $\alpha^\sharp$ is orthogonal to $D^\perp$ and*

$$(5.2.5) \qquad A_{JZ}X = g(J\alpha^\sharp, Z)X - (g(J\alpha^\sharp, X) + (JX\mu))Z$$

for each $X \in D$, $Z \in D^\perp$, and for smooth function μ on M with $W\mu = 0$ for each $W \in D^\perp$.

Proof. Let $M = N_T \times_f N^\perp$ be a CR-warped product submanifold of $\bar{M}$. Using Proposition 1.2.1 follows

$$0 = g(J\bar{\nabla}_X Z, JY) = g(\bar{\nabla}_X JZ, JY) - g((\bar{\nabla}_X J)Z, JY),$$

for each $X, Y \in TN_T$ and $Z, W \in TN^\perp$. Thus by (1.3.14) and Weingarten formula,

$$(5.2.6) \qquad g(A_{JZ}X,Y) = g(J\alpha^\sharp, Z)g(X,Y) + g(\alpha^\sharp, Z)g(JX,Y).$$

Furthermore, we have,

$$\begin{aligned}(Xlnf)g(Z,W) &= g(J\bar{\nabla}_Z X, JW)\\ &= g(\bar{\nabla}_Z JX, JW) - g((\bar{\nabla}_Z J)X, JW)\\ &= g(A_{JW}JX, Z) + g(\alpha^\sharp, X)g(Z,W).\end{aligned}$$

Therefore,

$$g(A_{JZ}X, W) = -(JXlnf + g(J\alpha^\sharp, X))g(Z,W). \tag{5.2.7}$$

Taking account of Proposition (5.2.1) while combining (5.2.6) and (5.2.7), we arrive at

$$A_{JZ}X = g(J\alpha^\sharp, Z)X - (JXlnf + g(J\alpha^\sharp, X))Z. \tag{5.2.8}$$

Conversely, suppose that M is a CR-submanifold of $\bar{M}$ with $\alpha^\sharp$ orthogonal to $D^\perp$ and satisfying (5.2.5) for some function μ on M, such that $W\mu = 0$ for all $W \in D^\perp$. Then,

$$g(h(X, JY) - h(Y, JX), FZ) = g(A_{FZ}X, JY) - g(A_{FZ}Y, JX).$$

By making use of (5.2.5), the right hand side reduces to $g(J\alpha^\sharp, Z)g(JX, Y)$, which on account of Proposition 5.1.2 shows that D is involutive. Moreover, since the totally real distribution is involutive (cf. Proposition 5.1.2), M is foliated by N_T and $N_\perp$.

Now, for any $X, Y \in D, Z \in D^\perp$, by (5.2.5) we might write

$$g(A_{JZ}X, Y) = g(J\alpha^\sharp, Z)g(X, Y).$$

By applying Weingarten formula, the above relation might be written as

$$g(\bar{\nabla}_X JZ, Y) + g(J\alpha^\sharp, Z)g(X, Y) = 0.$$

It can further be simplified to yield

$$g(Z, \nabla_X JY) - g(Z, (\bar{\nabla}_X J)Y) + g(J\alpha^\sharp, Z)g(X, Y) = 0.$$

which, by using (1.3.14), takes the form

$$g(\nabla_X JY, Z) = g(X, JY)g(\alpha^\sharp, Z).$$

As $\alpha^\sharp$ is orthogonal to $D^\perp$, it follows that the leaves of D are totally geodesic in M.

Let $N_\perp$ be a leaf of $D^\perp$, ∇' be the Levi-Civita connection on $N_\perp$, and h^0 be the second fundamental form of $N_\perp$ into M. Then by Gauss formula,

$$0 = g(\nabla'_Z W, X) = g(\nabla_Z W - h^0(Z, W), X).$$

for any $X \in D$ and $Z, W \in D^\perp$. That means

$$\begin{aligned} g(h^0(Z, W), X) &= g(\nabla_Z W, X) \\ &= g(\bar{\nabla}_Z W, X) \\ &= g(\bar{\nabla}_Z JW - (\bar{\nabla}_Z J)W, JX). \end{aligned}$$

By using (1.3.14) and Weingarten formula, the above equation takes the form

$$g(h^0(Z, W), X) = -g(A_{JW} JX, Z) - g(Z, W) g(\alpha^\sharp, X).$$

Now, on account of formula (5.2.5) the above equation can further be simplified to yield

$$(5.2.9), \qquad h^0(Z, W) = -g(Z, W)\nabla\mu$$

where $\nabla\mu$ is the gradient of μ. The above relation shows that the leaves of $D^\perp$ are totally umbilical in M with mean curvature vector μ. Moreover, the condition $Z\alpha = 0$ for all $Z \in D^\perp$ implies that the mean curvature vector is parallel. That is, the leaves of D^0 are extrinsic spheres in M. Hence, by Theorem 1.2.2, we get that M is locally a CR-warped product submanifold $N_T \times_\mu N_\perp$ of $\bar{M}$. This proves the theorem completely.

5.3. Inequalities for the length of the Second Fundamental Form

In this section, we calculate the length $\|h\|$ of the second fundamental form h of a warped product CR-submanifold $M_1 = N_\perp \times_f N_T$ and a CR-warped product $M_2 = N_T \times_f N_\perp$ of an l.c.K. manifold $\bar{M}$ under the assumption that the Lee-vector field $\alpha^\sharp$ is tangent to M_2.

Now, we put dim $\bar{M} = 2m$, dim M_1= dim $M_2 = n$, dim $M_T = 2p$ and dim $M_\perp = q$ $(2p + q = n)$.

Let $\{e_1, ..., e_p, e_1^*, ..., e_p^*\}$, $\{e_{2p+1}, ..., e_n\}$, $\{e_{2p+1}^*, ..., e_n^*\}$ and $\{e_{n+p+1}, ..., e_{2m}\}$ be the local orthonormal basis of $D, D^\perp, JD^\perp$ and μ respectively, where $e_i^* = Je_i$ for $i \in \{1, ..., p\}$ and $e_{2p+i}^* = Je_{2p+i}$ for $i \in \{1, ..., p\}$.

Remark 5.3.1. It is known that the dimension of the holomorphic submanifold is even.

The length $\|h\|$ of the second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 \tag{5.3.1}$$

The above (5.3.1) is written as

$$\begin{aligned}
\|h\|^2 &= \sum_{r=n+1}^{n+q} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+q+1}^{2m} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 \\
&\geq \sum_{r=n+1}^{n+q} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 = \sum_{l=2q+1}^{n} \sum_{i,j=1}^{n} g(h(e_i, e_j), Je_l)^2 \\
&= \sum_{l=2q+1}^{n} \left\{ \sum_{i,j=1}^{2q} g(h(e_i, e_j), e_l^*)^2 + 2 \sum_{i=1}^{2q} \sum_{j=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 \right. \\
&\quad \left. + \sum_{i,j=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 \right\}
\end{aligned}$$

Let M_1 be a non-trivial proper warped product CR-submanifold in an l.c.K. manifold $\bar{M}$. Then we know that

$$Zlnf = g(\alpha^\sharp, Z) \tag{5.3.2}$$

for any $Z \in D^\perp$. On a CR-submanifold M of an l.c.K. manifold $\bar{M}$, we know the following relation are known [42],[43]

$$\begin{aligned}
g(\nabla_U Z, X) &= g(JA_{JZ}U, X) + g(\alpha^\sharp, Z)g(U, X) \\
&\quad + g(U, Z)g(\alpha^\sharp, X) - g(\beta^\sharp, Z)g(JU, X)
\end{aligned} \tag{5.3.3}$$

By virtue of (5.3.3) and (5.3.2), we have

$$g(h(X, Y), JZ) = g(\beta^\sharp, Z)g(X, Y) \tag{5.3.4}$$

for any $X, Y \in D$ and $Z \in D^\perp$. The equation (5.3.4) means

$$g(h(e_i, e_j), e_l^*) = g(\beta^\sharp, e_l)\delta ij$$

for $i, j \in \{1, ..., 2q\}$ and $l \in \{2q+1, ..., n\}$. So, we get

$$\sum_{l=2q+1}^{n} \sum_{i,j=1}^{2q} g(h(e_i, e_j), e_l^*)^2 = 2q \sum_{l=2q+1}^{n} g(\beta^\sharp, e_l)^2 \tag{5.3.5}$$

Next we have from (5.1.4)

$$g(h(X,Z),JW)=g(\beta^\sharp,X)g(Z,W) \tag{5.3.6}$$

for any $X\in D$ and $Z,W\in D^\perp$. Using (5.3.6) we obtain

$$g(h(e_i,e_j),Je_k)=g(\beta^\sharp,e_i)\delta_{jk}$$

for $i\in\{1,....,2q\}$ and $j,k\in\{2q+1,....,n\}$. Thus we have

$$\begin{aligned}\sum_{i=1}^{2q}\sum_{j,k=2q+1}^{n} g(h(e_i,e_j),e_k^*)^2 &=\sum_{i=1}^{2q}\sum_{j,k=2q+1}^{n}\{g(\beta^\sharp,e_i)\delta_{jk}\}^2\\ &=q\sum_{i=1}^{2q} g(\beta^\sharp,e_i)^2\end{aligned} \tag{5.3.7}$$

By virtue of (5.3.5) and (5.3.7)

$$\|h\|^2\geq 2q\sum_{l=2q+1}^{n} g(\beta^\sharp,e_l)^2+2q\sum_{i=1}^{2q} g(\beta^\sharp,e_i)^2+\sum_{i,j,l=2q+1}^{n} g(h(e_i,e_j),e_l^*)^2. \tag{5.3.8}$$

Thus we have

Theorem 5.3.1. [9] *The length $\|h\|$ of the second fundamental form h of a non-trivial proper warped product CR-submanifold M_1 in an l.c.K. manifold $\bar{M}$ satisfies*

$$\|h\|^2\geq 2\{q\sum_{l=2q+1}^{n} g(\beta^\sharp,e_l)^2+q\sum_{i=1}^{2q} g(\beta^\sharp,e_i)^2\}. \tag{5.3.9}$$

The equality in (5.3.9) *is satisfied if and only if*

$$g(h(TM,TM),\mu)=\{0\},\qquad g(h(D^\perp,D^\perp),JD^\perp)=\{0\}. \tag{5.3.10}$$

Next, let M_2 be a proper CR-warped product in an l.c.K. manifold $\bar{M}$ and the Lee-vector field $\alpha^\sharp$ be tangent to M_2.

Remark 5.3.2. By Proposition 5.2.1, in our case, the Lee-vector field $\alpha^\sharp$ is in D.

On the other hand, we have from Proposition 5.2.1 and (5.2.1)

$$g(h(e_i,e_j),e_l^*)=g(e_i,e_j)g(\beta^\sharp,e_l)=-\delta_{ij}g(\alpha^\sharp,e_l^*)=0$$

for $i, j \in \{1, ..., 2q\}$ and $l \in \{2q+1, ..., n\}$. This means that

$$g(h(D,D), JD^{\perp}) = 0. \tag{5.3.11}$$

So we have

$$\begin{aligned}
\|h\|^2 &\geq \sum_{l=2q+1}^{n} \left\{ 2\sum_{i=1}^{2q} \sum_{j=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 + \sum_{i,j=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 \right\} \\
&= 2\sum_{i=1}^{2q} \sum_{j,l=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 + \sum_{i,j,l=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2
\end{aligned}$$

Next by virtue of (5.2.3), we get

$$g(h(e_i, e_j), e_l^*) = -(Je_i \, lnf + g(\alpha^\sharp, Je_i))\delta_{jl}$$

for $i \in \{1, ..., 2q\}$ and $j, l \in \{2q+1, ..., n\}$. So, we have

$$\sum_{j,l=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 = (Je_i \, lnf - g(\beta^\sharp, e_i))^2 q \tag{5.3.12}$$

Thus we have from (5.3.12)

$$\begin{aligned}
\|h\|^2 &\geq 2\sum_{i=1}^{2q} \{(Je_i \, lnf - g(\alpha^\sharp, Je_i))^2 q\} + \sum_{i,j,l=2q+1}^{n} g(h(e_i, e_j), e_l^*)^2 \\
&\geq 2q \sum_{i=1}^{2q} (Je_i \, lnf - g(\alpha^\sharp, Je_i))^2.
\end{aligned}$$

Thus we have

$$\|h\|^2 \geq 2q \sum_{i=1}^{2q} (Je_i \, lnf - g(\alpha^\sharp, Je_i))^2. \tag{5.3.13}$$

Next, we consider the equality case of (5.3.13). Then we have

$$g(h(TM, TM), \mu) = \{0\}, \quad g(h(D^{\perp}, D^{\perp}), JD^{\perp}) = \{0\}.$$

By virtue of (5.3.11) and the above relations, we get

$$h(D, D) = 0, \quad h(D^{\perp}, D^{\perp}) = 0, \quad h(D, D^{\perp}) \subset JD^{\perp}. \tag{5.3.14}$$

Hence, we have

Theorem 5.3.2. [9] *Let $M_2 = N_T \times_f N_\perp$ be a proper CR-warped product in an l.c.K. manifold $\bar{M}$. If the Lee-vector field $\alpha^\sharp$ is tangent to M_2, the length $\|h\|$ of the second fundamental form h satisfies* (5.3.13). *And the equality of* (5.3.13) *is satisfied if and only if the second fundamental form h satisfies* (5.3.14).

5.4. Generic Warped Product Submanifolds

To extend the study of CR-submanifolds as warped product submanifolds of Kaehler and l.c.K. manifolds, we now consider generic submanifolds immersed as warped product submanifolds in an l.c.K. manifold $\bar{M}$. Thus, from now on, we assume that N_1, N_2 are non-trivial submanifolds of $\bar{M}$, such that $M = N_1 \times_f N_2$ is a warped product submanifold of $\bar{M}$. If one of the factors of M is a holomorphic submanifold, then M is called a generic warped product submanifold of $\bar{M}$. CR-warped product and warped product CR-submanifolds are thus special cases of generic warped product submanifolds. We begin the section with proving some relevant formulae for generic warped product submanifolds of an l.c.K. manifold.

Proposition 5.4.1. [32] *Let $\bar{M}$ be an l.c.K. manifold and $N_1 \times_f N_2$ be a generic warped product submanifold of $\bar{M}$. Then,*

$$g(h(X,Z),FW) = g(h(X,W),FZ), \tag{5.4.1}$$

$$h_{FD^0}(X,Y) = -g(X,Y)\alpha^\sharp_{FD^0} \tag{5.4.2}$$

for each vector fields X, Y on the holomorphic submanifold and Z, W on the other factor of M.

Proof. By Gauss formula,

$$\begin{aligned} g(h(X,Z),FW) &= g(\bar{\nabla}_X Z, FW) \\ &= -g(J\bar{\nabla}_X Z, W) - g(\nabla_X Z, PW) \\ &= g((\bar{\nabla}_X J)Z, W) - g(\bar{\nabla}_X JZ, W) - g(\nabla_X Z, PW) \\ &= g((\bar{\nabla}_X J)Z, W) - g(\bar{\nabla}_X PZ, W) - g(\bar{\nabla}_X FZ, W) - g(\nabla_X Z, PW). \end{aligned}$$

The first term on the right hand side of above equation is zero by (1.3.14) whereas by making use of proposition 1.2.1 and Weingarten formula, the other terms, in both cases, namely when N_1 and N_2 are holomorphic, are reduced to $g(A_{FZ}X, W)$. This proves (5.4.1).

To prove (5.4.2), we consider (1.3.14) which implies that

$$g((\bar{\nabla}_X J)Y, Z) = g(JX, Y)g(\alpha^\sharp, Z) + g(X, Y)g(J\alpha^\sharp, Z).$$

Simplifying the left hand side, the above equation takes the form

$$g(h(X,Y), FZ) = g(JY, \nabla_X Z) + g(Y, \nabla_X PZ) + g(X,Y)g(J\alpha^\sharp, Z) + g(JX, Y)g(\alpha^\sharp, Z) \tag{5.4.3}$$

Case (i) When N_1 is holomorphic, then by using Proposition 1.2.1, the above equation yields

$$g(h(X,Y), FZ) = g(JX, Y)g(\alpha^\sharp, Z) + g(X, Y)g(J\alpha^\sharp, Z) \tag{5.4.4}$$

from which one might drive

$$g(h(X,Y), FZ) = -g(X,Y)g(\alpha^\sharp, JZ) \tag{5.4.5}$$

and

$$g(JX, Y)g(\alpha^\sharp, Z) = 0. \tag{5.4.6}$$

It follows from (5.4.6) that either the holomorphic submanifold N_1 is trivial or the Lee-vector field is orthogonal to the submanifold N_2.

On account of the above observation in (5.4.5), it follows that, on a non-trivial generic warped product submanifold of an l.c.K. manifold,

$$g(h(X,Y), FZ) = -g(X,Y)g(\alpha^\sharp, FZ). \tag{5.4.7}$$

Case (ii) When N_2 is holomorphic, then by using Proposition 1.2.1, equation (5.4.3) yields

$$g(h(X,Y), FZ) = (Zlnf - g(\alpha^\sharp, Z))g(X, JY) + (PZlnf - g(\alpha^\sharp, JZ))g(X,Y). \tag{5.4.8}$$

Comparing symmetric and skew-symmetric parts in the last equation, we obtain

$$g(h(X,Y), FZ) = (PZlnf - g(\alpha^\sharp, JZ))g(X,Y), \tag{5.4.9}$$

and

$$Zlnf = g(\alpha^\sharp, Z). \tag{5.4.10}$$

On account of (5.4.10), we deduce that

$$g(h(X,Y),FZ) = -g(\alpha^\sharp, FZ))g(X,Y), \tag{5.4.11}$$

Thus (5.4.2) is proved in view of (5.4.7) and (5.4.11).

Theorem 5.4.1. [32] *Let $M = N_1 \times_f N_2$ be a generic warped product submanifold of an l.c.K. manifold $\bar{M}$. Then*

$$U_1 lnf = g(\alpha^\sharp, U_1) \tag{5.4.12}$$

for each $U_1 \in TN_1$, or else $M$is a CR-warped product submanifold.

Proof. If N_1 is a holomorphic submanifold, then for $U_1 \in TN_1$ and $U_2, V_2 \in TN_2$, we might write

$$\begin{aligned} g(h(JU_1,U_2),FV_2) &= g(\bar{\nabla}_{U_2} JU_1, FV_2) \\ &= g((\bar{\nabla}_{U_2} J)U_1, JV_2) + g(\bar{\nabla}_{U_2} U_1, V_2) - g(\nabla_{U_2} JU_1, PV_2). \end{aligned}$$

The right hand side, on account of (1.3.14) takes form

$$-g(\alpha^\sharp, U_1)g(U_2,V_2) - g(J\alpha^\sharp, U_1)g(U_2, PV_2) + g(\nabla_{U_2} U_1, V_2) - g(\nabla_{U_2} JU_1, PV_2).$$

which, by making use of Proposition 1.2.1, is reduced to

$$(U_1 lnf - g(\alpha^\sharp, U_1))g(U_2,V_2) + (g(\alpha^\sharp, JU_1) - JU_1 lnf)g(U_2, PV_2).$$

Thus,

$$\begin{aligned} g(h(U_1,U_2),FV_2) &= (g(\alpha^\sharp, JU_1) - JU_1 lnf)g(U_2,V_2) \\ &\quad + (U_1 lnf - g(\alpha^\sharp, U_1))g(PU_2,V_2). \end{aligned} \tag{5.4.13}$$

By making use of (5.4.1), it can be deduced from (5.4.13) that

$$g(h(U_1,U_2),FV_2) = (g(\alpha^\sharp, JU_1) - JU_1 lnf)g(U_2,V_2) \tag{5.4.14}$$

and

$$(U_1 lnf - g(\alpha^\sharp, U_1))g(PU_2,V_2) = 0. \tag{5.4.15}$$

It follows from (5.4.15) that either M is a CR-warped product submanifold or

$$U_1 lnf = g(\alpha^\sharp, U_1).$$

In contrast, if N_2 is holomorphic, then (5.4.12) is proved by virtue of (5.4.10). This proves the assertion completely.

As an immediate consequence of Theorem 5.4.1, we might state

Corollary 5.4.1. *If the Lee-vector field $\alpha^\sharp$ is orthogonal to the first factor, then a proper generic warped product submanifold $N_1 \times_f N_2$ of an l.c.K. manifold is a generic product, i.e., a trivial warped product submanifold.*

Thus, a proper generic warped product submanifold might be found in an l.c.K. manifold if the Lee-vector field has a non-zero component along the first factor.

Example 5.4.1. [32] Let $\bar{M} = S^1 \times S^7$ and Ξ =span $\{e_1, e_2, ..., e_5\}$. Put $S^4 = S^7 \cap \Xi$ and $M_2^4 = \{x = \sum_{i=1}^{4} x_i e_i \in S^4 : 0 < x_5 < 1\}$. For each point $x \in M_2^4$, let D'_x be the subspace of $T_x M_2^4$ defined by $D'_x = \{U \in T_x M_2^4 : (U, J_0 x) = 0, (U, e_5) = 0\}$.

For each point $(e^{it}, x) \in S^1 \times M_2^4$, let $D(e^{it}, x)$ be the subspace of $T(e^{it}, x)M$ defined by

$$D(e^{it}, x) = \{(0, U) \in T(e^{it}, x)M : U \in D'_x\}.$$

Then, we might easily observe that $M = S^1 \times_{e^{it}} M_2^4$ is a proper generic warped product submanifold of $\bar{M}$ with the holomorphic distribution D.

In particular, for generic warped product submanifolds of Kaehler manifolds, we conclude,

Corollary 5.4.2. *There does not exist a generic warped product submanifold in a Kaehler manifold with first factor to be a holomorphic submanifold unless the second factor is totally real. Also, there does not exist a generic warped product submanifold in a Kaehler manifold with second factor to be a holomorphic submanifold, i.e., the only generic warped product submanifolds in a Kaehler manifold are CR-warped product submanifolds.*

Note: Corollary 5.4.2 generalizes the result of B. Sahin [49] concerning the non-existent of warped product semi-slant submanifolds in Kaehler manifolds.

Theorem 5.4.1 ensures the existence of generic warped product submanifolds in l.c.K. manifolds with $\alpha^\sharp$ having a non-zero component along the first factor. The statement can further be extended as:

Theorem 5.4.2. [32] *Let $M = N_1 \times_f N_2$ be a generic warped product submanifold of an l.c.K. manifold $\bar{M}$. Then the Lee-vector field is orthogonal* *to the second factor* N_2.

Proof. Case(i) Let N_1 be holomorphic. In this case the assertion directly follows from (5.4.6).

Case (ii) Let N_1 be holomorphic. Then, by Gauss formula, we might write

$$g(h(U_1, U_2), FV_1) = g(\bar{\nabla}_{U_1} U_2, FV_1).$$

for each $U_1, V_1 \in TN_1$ and $U_2 \in TN_2$. By applying Proposition 1.2.1, the right hand side of the above equation is reduced to $g((\bar{\nabla}_{U_1} J)U_2, V_1)$. Now, by using (1.3.14), the equation takes the form

$$g(h(U_1, U_2), FV_1) = g(\alpha^\sharp, JU_2)g(U_1, V_1) - g(\alpha^\sharp, U_2)g(U_1, PV_1). \tag{5.4.16}$$

Making use of (5.4.1), we deduce from (5.4.16) that

$$g(h(U_1, U_2), FV_1) = g(\alpha^\sharp, JU_2)g(U_1, V_1) \tag{5.4.17}$$

and

$$g(\alpha^\sharp, U_2)g(PU_1, V_1) = 0 \tag{5.4.18}$$

If M is proper, then (5.4.18) implies that $\alpha^\sharp$ is orthogonal to N_2. In contrast, if M is a warped product CR-submanifold, then the assertion follows from Proposition 5.2.1. This proves the theorem.

Corollary 5.4.3. *A generic warped product submanifold* M *of an l.c.K. manifold* $\bar{M}$ *is trivial if and only if the Lee-vector field is normal to the submanifold* M.

The proof follows from Theorem 5.4.2 and Corollary 5.4.1.

Note: The above corollary is an extension of the condition for CR-submanifold to be a CR-product in an l.c.K. manifold (cf. [8]).

Corollary 5.4.4. *Let* M *be a proper generic submanifold immersed as warped product submanifold in an l.c.K. manifold* $\bar{M}$. *Then* $h(U_1, U_2) \in \mu$ *for each* $U_1 \in TN_1$ *and* $U_2 \in TN_2$.

Proof. When N_1 is holomorphic submanifold, the proof follows on account of (5.4.12) in (5.4.14). In contrast, when N_2 is holomorphic, then by (5.4.17)

$$g(h(U_1, U_2), FV_1) = g(\alpha^\sharp, JU_2)g(U_1, V_1)$$

The proof follows from the above equation on account of Theorem 5.4.2.

If N_T and N^0 denote the holomorphic and purely real submanifolds of an l.c.K. manifold $\bar{M}$, then the two cases of generic warped product submanifolds are $N^0 \times_f N_T$ and $N_T \times_f N^0$. Let dim $(\bar{M}) = 2m$, dim $(M) = n$, dim$(N_T) = 2p$, dim $(N^0) = q$ i.e., $2p + q = n$. Further, let $\{e_1, e_2, ..., e_p, e_{p+1} = Je_1, ..., e_{2p} = Je_p\}, \{e_{2p+1}, ..., e_n\}, \{E_1, E_2, ...E_q\}$ and $\{E_{q+1}, E_{q+2}, ..., E_{2m}\}$ be the local orthonormal frames on $T(N_T), T(N^0), F(TN^0)$ and μ, respectively.

Theorem 5.4.3. [32]. *Let $M = N_1 \times_f N_2$ be a generic warped product submanifold of an l.c.K. manifold $\bar{M}$. Then the squared norm of the second fundamental form h of the immersion of M into $\bar{M}$ satisfies*

$$\|h\|^2 \geq 2[p\|\alpha^\sharp_{F(TN^0)}\|^2 + \sum_{i=1}^{2p} q(g(\alpha^\sharp, Je_i) - Je_i lnf)^2], \tag{5.4.19}$$

if the first factor of M is holomorphic submanifold of $\bar{M}$, and

$$\|h\|^2 \geq 2p\|\alpha^\sharp_{F(TN^0)}\|^2 \tag{5.4.20}$$

if the second factor of M is holomorphic in $\bar{M}$, where $\alpha^\sharp_{F(TN^0)}$ denotes the component of $\alpha^\sharp$ in $F(TN^0)$.

Proof. The squared norm of the second fundamental form of M into $\bar{M}$ is defined by

$$\|h\|^2 = \sum_{r=1}^{2m-n} \sum_{i,j=1}^{n} g(h(e_i, e_j), E_r)^2.$$

It can be expanded as

$$\begin{aligned}\|h\|^2 &= \sum_{k=1}^{q} \sum_{i,j=1}^{n} g(h(e_i, e_j), Fe_{2p+k})^2 + \sum_{r=q+1}^{2m-q} \sum_{i,j=1}^{n} g(h(e_i, e_j), E_r)^2 \\ &\geq \sum_{k=1}^{q} \sum_{i,j=1}^{n} g(h(e_i, e_j), Fe_{2p+k})^2\end{aligned}$$

$$\|h\|^2 \geq \sum_{k=1}^{q} [\sum_{i,j=1}^{2p} g(h(e_i, e_j), Fe_{2p+k})^2 \tag{5.4.21}$$

$$+\sum_{i=1}^{2p}\sum_{j=2p+1}^{2p+q} g(h(e_i,e_j),Fe_{2p+k})^2].$$

Case (i) When N_1 is holomorphic submanifold of $\bar{M}$. In this case, by virtue of formula (5.4.2), the first term on the right hand side of inequality (5.4.21) reduces to $2p\|\alpha^\sharp_{F(TN^0)}\|^2$ whereas the second term, by virtue of (5.4.14), is reduced to $\sum q(g(\alpha^\sharp, Je_i) - Je_i lnf)^2$. Therefore inequality (5.4.21) in this case takes the form

$$\|h\|^2 \geq 2[p\|\alpha^\sharp_{F(TN^0)}\|^2 + \sum_{i=1}^{2p} q(g(\alpha^\sharp, Je_i) - Je_i lnf)^2].$$

Case (ii) When N_2 is holomorphic submanifold of $\bar{M}$. In this case, the assertion follows from inequality (5.4.21) on account of Corollary 5.4.4 and formula (5.4.2).

The equality in (5.4.19) and (5.4.20) holds if the purely real distribution is totally geodesic in $\bar{M}$ and $h(U,V)$ lies in $F(D^0)$ for each $U,V \in D^0$.

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